

S. Pilipović* - N. Teofanov* - J. Toft

WAVE-FRONT SETS IN FOURIER LEBESGUE SPACES

Abstract. We consider wave front sets in the framework of weighted Fourier Lebesgue spaces, \mathcal{FL}_s^q . We prove that

$$WF_{\mathcal{FL}_{s-m}^q}(Af) \subset WF_{\mathcal{FL}_s^q}(f) \subset WF_{\mathcal{FL}_{s-m}^q}(Af) \cup \text{char}A$$

where A is properly supported pseudodifferential operator of order m and $\text{char}A$ denotes the set of characteristic points of A .

1. Introduction.

In the present paper, we study certain aspects of microlocal analysis in Fourier Lebesgue spaces. More precisely, we define wave front sets in those spaces and show that usual mapping properties for a class of pseudo-differential operators which are valid for classical wave front sets (cf. [3, Chapter VIII]) also hold for our wave front sets. We refer to [5, 6] for detailed exposition of the theory. The interest in Fourier Lebesgue spaces increased with the recent study of pseudo-differential and Fourier integral operators in Fourier Lebesgue spaces and their connection to modulation spaces in different contexts [1, 4, 7, 8].

The paper is organized as follows. In Section 2 we fix some notions and notation. The precise definition of Fourier Lebesgue spaces and wave front sets in the context of Fourier Lebesgue spaces is given in Section 3. In the same section we establish basic properties of the introduced wave front, Propositions 1, 2 and 3. In Section 4 we study the continuity properties of pseudo-differential operators of the Hörmander class S^m on Fourier Lebesgue spaces, Propositions 4 and 5. Section 5 is devoted to the study of microlocal properties of localized version of pseudo-differential operators, Theorems 1 and 2. As a corollary, we discuss the relationship between our wave front sets and the classical ones. In the last section we introduce wave front sets in modulation spaces and show that Fourier Lebesgue spaces and modulation spaces are locally the same.

*This research was supported by Ministry of Science of Serbia, project no. 144016.

2. Notions and notation.

By Γ we denote an open cone in $\mathbf{R}^d \setminus 0$ and by X we always denote an open set in \mathbf{R}^d . A conic neighborhood of a point $(x_0, \xi_0) \in \mathbf{R}^d \times \mathbf{R}^d \setminus 0$ is a product $X \times \Gamma$, where X is a neighborhood of x_0 in \mathbf{R}^d and Γ is an open cone in \mathbf{R}^d which contains ξ_0 . Sometimes such cone is denoted by Γ_{ξ_0} . When $x, \xi \in \mathbf{R}^d$, their scalar product is simply denoted by $x\xi$. As usual, $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

Next we discuss appropriate conditions for the involved weight functions. Assume that ω and v are positive and measurable functions on \mathbf{R}^d . Then ω is called v -moderate weight if

$$(2.1) \quad \omega(x+y) \leq C\omega(x)v(y)$$

for some constant C which is independent of $x, y \in \mathbf{R}^d$. If v in (2.1) can be chosen as a polynomial, then ω is called polynomially moderated. We let $\mathcal{P}(\mathbf{R}^d)$ be the set of all polynomially moderated functions on \mathbf{R}^d .

The Fourier transform \mathcal{F} is the linear and continuous mapping on $\mathcal{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int f(x)e^{-ix\xi} dx, \quad \xi \in \mathbf{R}^d,$$

when $f \in L^1(\mathbf{R}^d)$. We recall that \mathcal{F} is a homeomorphism on $\mathcal{S}'(\mathbf{R}^d)$ which restricts to a homeomorphism on $\mathcal{S}(\mathbf{R}^d)$ and to a unitary operator on $L^2(\mathbf{R}^d)$.

We say that a distribution $f \in \mathcal{D}'(\mathbf{R}^d)$ is *microlocally smooth* at $(x_0, \xi_0) \in \mathbf{R}^d \times \mathbf{R}^d \setminus 0$ if there exists $\varphi \in C_0^\infty(\mathbf{R}^d)$, $\varphi(x_0) \neq 0$, and an open cone Γ_{ξ_0} such that for every $N \in \mathbf{N}$, there exists $C_N > 0$, such that $|\widehat{f\varphi}(\xi)| \leq C_N \langle \xi \rangle^{-N/2}$, $\xi \in \Gamma_{\xi_0}$. The *wave front set* of f , $WF(f)$ is the complement of the set of points (x_0, ξ_0) where f is microlocally smooth.

For $q \in [1, \infty]$ we let $q' \in [1, \infty]$ denote the conjugate exponent, i. e. $1/q + 1/q' = 1$.

3. Wave front sets in Fourier Lebesgue spaces

DEFINITION 1. *Let there be given $\omega \in \mathcal{P}(\mathbf{R}^{2d})$ and let $q \in [1, \infty]$. The (weighted) Fourier-Lebesgue space $\mathcal{FL}_{(\omega)}^q(\mathbf{R}^d)$ is the Banach space which consists of all $f \in \mathcal{S}'(\mathbf{R}^d)$ such that*

$$(3.1) \quad \|f\|_{\mathcal{FL}_{(\omega)}^q} = \|f\|_{\mathcal{FL}_{(\omega),x}^q} \equiv \|\widehat{f} \cdot \omega(x, \cdot)\|_{L^q} < \infty.$$

We say that $f \in \mathcal{D}'(X)$ is locally in $\mathcal{FL}_{(\omega)}^q(\mathbf{R}^d)$, if $\chi f \in \mathcal{FL}_{(\omega)}^q(\mathbf{R}^d)$ for every $\chi \in C_0^\infty(X)$ and in that case we use the notation $f \in \mathcal{FL}_{(\omega),loc}^q(\mathbf{R}^d) = \mathcal{FL}_{(\omega),loc}^q(X)$. It is said that $f \in \mathcal{FL}_{(\omega),loc}^q(X)$ at x_0 if there exists a function $\chi \in C_0^\infty(\mathbf{R}^d)$, $\chi(x_0) \neq 0$, such that $\chi f \in \mathcal{FL}_{(\omega)}^q(X)$.

Weights $\omega(x, \xi)$ in (3.1) depend on both x and ξ , although $\widehat{f}(\xi)$ only depends on ξ . However, since ω is v -moderate for some $v \in \mathcal{P}(\mathbf{R}^{2d})$, different choices of x give rise to equivalent norms. Therefore, the condition $\|f\|_{\mathcal{F}L_{(\omega),x}^q} < \infty$ is independent of x .

In the remaining part of the paper we study weighted Fourier-Lebesgue spaces with weights which depend on ξ . Thus, with $\omega_0(\xi) = \omega(0, \xi) \in \mathcal{P}(\mathbf{R}^d)$,

$$f \in \mathcal{F}L_{(\omega)}^q(\mathbf{R}^d) = \mathcal{F}L_{(\omega_0)}^q(\mathbf{R}^d) \iff \|f\|_{\mathcal{F}L_{(\omega_0)}^q} \equiv \|\widehat{f}\omega_0\|_{L^q} < \infty.$$

We usually assume that the involved weight functions $\omega_0(\xi)$ is given by $\omega_0(\xi) = \omega(x_0, \xi) = \langle \xi \rangle^s = (1 + |\xi|^2)^{s/2}$, for some $x_0 \in \mathbf{R}^d$ and $s \in \mathbf{R}$. In this case we use the notation $\mathcal{F}L_s^q(\mathbf{R}^d)$ instead of $\mathcal{F}L_{(\omega_0)}^q(\mathbf{R}^d)$. If $\omega = 1$, then the notation $\mathcal{F}L^q(\mathbf{R}^d)$ is used instead of $\mathcal{F}L_{(\omega)}^q(\mathbf{R}^d)$.

Assume that $\omega_0 \in \mathcal{P}(\mathbf{R}^d)$, an open cone $\Gamma \subseteq \mathbf{R}^d \setminus 0$ and $q \in [1, \infty]$ are fixed. For any $f \in \mathcal{S}'(\mathbf{R}^d)$, let

$$|f|_{\mathcal{F}L_{(\omega_0)}^{q,\Gamma}} \equiv \left(\int_{\Gamma} |\widehat{f}(\xi)\omega_0(\xi)|^q d\xi \right)^{1/q}$$

(with obvious interpretation when $q = \infty$). We note that $|\cdot|_{\mathcal{F}L_{(\omega_0)}^{q,\Gamma}}$ defines a semi-norm on $\mathcal{S}'(\mathbf{R}^d)$ which might attain the value $+\infty$.

We let $\Theta_{\mathcal{F}L_{(\omega_0)}^q}(f)$ to be the set of all $\xi \in \mathbf{R}^d \setminus 0$ such that $|f|_{\mathcal{F}L_{(\omega_0)}^{q,\Gamma}} < \infty$ for some $\Gamma = \Gamma_{\xi}$. Its complement in $\mathbf{R}^d \setminus 0$ is denoted by $\Sigma_{\mathcal{F}L_{(\omega_0)}^q}(f)$.

We have now the following result.

PROPOSITION 1. *Assume that $q \in [1, \infty]$, $\chi \in \mathcal{S}'(\mathbf{R}^d)$, and that $\omega_0 \in \mathcal{P}(\mathbf{R}^d)$. Also assume that $f \in \mathcal{E}'(\mathbf{R}^d)$. Then*

$$(3.2) \quad \Sigma_{\mathcal{F}L_{(\omega_0)}^q}(\chi f) \subseteq \Sigma_{\mathcal{F}L_{(\omega_0)}^q}(f).$$

Proof. Assume that $\xi_0 \in \Theta_{\mathcal{F}L_{(\omega_0)}^q}(f)$, and choose open cones Γ_1 and Γ_2 in \mathbf{R}^d such that $\overline{\Gamma_2} \subseteq \Gamma_1$. Since f has compact support, it follows that $|\widehat{f}(\xi)\omega_0(\xi)| \leq C\langle \xi \rangle^{N_0}$ for some positive constants C and N_0 . The idea of the proof is to show that for each N , there are constants C_N such that

$$(3.3) \quad |\chi f|_{\mathcal{F}L_{(\omega_0)}^{q_2,\Gamma_2}} \leq C_N \left(|f|_{\mathcal{F}L_{(\omega_0)}^{q_1,\Gamma_1}} + \sup_{\xi \in \mathbf{R}^d} (|\widehat{f}(\xi)\omega_0(\xi)|\langle \xi \rangle^{-N}) \right)$$

$$\text{when } q_1 \leq q_2, \quad \overline{\Gamma_2} \subseteq \Gamma_1 \quad \text{and} \quad N = 1, 2, \dots$$

The result then follows by taking $q_1 = q_2 = q$ and $N \geq N_0$. We refer to [5] for details of the proof of (3.3). \square

Now we are ready to define wave front sets in the framework of Fourier Lebesgue spaces.

DEFINITION 2. Assume that $q \in [1, \infty]$, $f \in \mathcal{D}'(\mathbf{R}^d)$ and $\omega_0 \in \mathcal{P}(\mathbf{R}^d)$. The wave-front set $WF_{\mathcal{F}L_{(\omega_0)}^q}(f)$ with respect to $\mathcal{F}L_{(\omega_0)}^q(\mathbf{R}^d)$ consists of all pairs (x_0, ξ_0) in $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_{\mathcal{F}L_{(\omega_0)}^q}(\chi f)$, holds for each $\chi \in C_0^\infty(\mathbf{R}^d)$ such that $\chi(x_0) \neq 0$.

The following proposition shows that the wave-front set $WF_{\mathcal{F}L_{(\omega_0)}^q}(f)$ decreases with respect to the parameter q and increases with respect to the weight function ω , when $f \in \mathcal{D}'(\mathbf{R}^d)$ is fixed.

PROPOSITION 2. Assume that $f \in \mathcal{D}'(\mathbf{R}^d)$, $q_j \in [1, \infty]$ and $\omega_j \in \mathcal{P}(\mathbf{R}^d)$ for $j = 1, 2$ satisfy

$$(3.4) \quad q_1 \leq q_2, \quad \text{and} \quad \omega_2(\xi) \leq C\omega_1(\xi),$$

for some constant C which is independent of $x, \xi \in \mathbf{R}^d$. Then

$$WF_{\mathcal{F}L_{(\omega_2)}^{q_2}}(f) \subseteq WF_{\mathcal{F}L_{(\omega_1)}^{q_1}}(f).$$

Proof. It is no restriction to assume that f has compact support, and that $\omega_1(\xi) = \omega_2(\xi) = \omega_0(\xi)$. This implies that

$$\sup_{\xi \in \mathbf{R}^d} |\langle \xi \rangle^{-N_0} \widehat{f}(\xi) \omega_0(\xi)| < \infty$$

provided N_0 is chosen large enough. Hence it follows from (3.3) that $\Theta_{\mathcal{F}L_{(\omega_1)}^{q_1}}(f) \subseteq \Theta_{\mathcal{F}L_{(\omega_2)}^{q_2}}(f)$, and the assertion follows. \square

PROPOSITION 3. Assume that $q \in [1, \infty]$, $f \in \mathcal{D}'(X)$, $\omega_0 \in \mathcal{P}(\mathbf{R}^d)$ and $(x_0, \xi_0) \in X \times (\mathbf{R}^d \setminus 0)$. The following conditions are equivalent.

(1) There exist $g \in \mathcal{F}L_{(\omega_0)}^q(\mathbf{R}^d)$ such that $(x_0, \xi_0) \notin WF(f - g)$.

(2) $(x_0, \xi_0) \notin WF_{\mathcal{F}L_{(\omega_0)}^q}(f)$.

Proof. Let us first show that $\varphi g \in \mathcal{F}L_{(\omega_0)}^q(\mathbf{R}^d)$ if $\varphi \in \mathcal{S}(\mathbf{R}^d)$ and $g \in \mathcal{F}L_{(\omega_0)}^q(\mathbf{R}^d)$.

Since $\widehat{g\varphi}(\xi) = (\widehat{g} * \widehat{\varphi})(\xi)$, for $\psi \in \mathcal{S}(\mathbf{R}^d)$ such that $\widehat{\psi} = \varphi$ we have

$$\begin{aligned} \|g\varphi\|_{\mathcal{F}L^q_{(\omega_0)}} &= \left(\int_{\mathbf{R}^d} \left| \int_{\mathbf{R}^d} \widehat{g}(\xi - t)\psi(t)dt \right|^q \omega_0(\xi)^q d\xi \right)^{1/q} \\ &\leq \left(\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |\widehat{g}(\xi - t)| |\psi(t)| dt \right)^q \omega_0(\xi)^q d\xi \right)^{1/q} \\ &\leq \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |\widehat{g}(\xi - t)|^q |\psi(t)|^q \omega_0(\xi)^q d\xi \right)^{1/q} dt \\ &\leq \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |\widehat{g}(\xi - t)|^q \omega_0(\xi - t)^q d\xi \right)^{1/q} |\psi(t)|^q v^q(t) dt \\ &\leq C(\psi) \|g\|_{\mathcal{F}L^q_{(\omega_0)}}, \end{aligned}$$

since v is of polynomial growth.

By the assumption, $f - g$ is microlocally smooth at (x_0, ξ_0) so there exists an open cone $\Gamma = \Gamma_{\xi_0}$ such that

$$(3.5) \quad \int_{\Gamma} |\widehat{\varphi f}(\xi)|^q \omega_0^q(\xi) d\xi < \infty,$$

where $\text{supp } \varphi$ can be chosen to be sufficiently close to x_0 . This, together with $\mathcal{F}\varphi f = \mathcal{F}(\varphi g) + \mathcal{F}(\varphi(f - g))$ implies that $(x_0, \xi_0) \notin WF_{\mathcal{F}L^q_{(\omega_0)}}(f)$.

Conversely, if $(x_0, \xi_0) \notin WF_{\mathcal{F}L^q_{(\omega_0)}}(f)$, then there exist $\varphi \in C_0^\infty$ such that $\varphi(x_0) \neq 0$ and a conic neighborhood Γ of ξ_0 such that (3.5) holds.

Let $g \in \mathcal{F}L^q_{(\omega_0)}(\mathbf{R}^d)$ be defined by

$$\widehat{g}(\xi) = \begin{cases} \widehat{\varphi f}(\xi), & \text{if } \xi \in \Gamma \\ 0, & \text{if } \xi \notin \Gamma. \end{cases}$$

Then $\widehat{h} = \widehat{\varphi f} - \widehat{g}$ vanishes in Γ and \widehat{h} has a polynomial bound. Therefore $(x_0, \xi_0) \notin WF(h)$. Choose $\psi \in C_0^\infty$ so that $\psi\varphi = 1$ in a neighborhood of x_0 . Note $(x_0, \xi_0) \notin WF(\psi h)$. Now, since

$$f - \psi g = (1 - \psi\varphi)f + \psi h,$$

we conclude that $(x_0, \xi_0) \notin WF(f - \psi g)$. Since $\psi g \in \mathcal{F}L^q_{(\omega_0)}(\mathbf{R}^d)$ the proof is complete. \square

4. Pseudo-differential operators

If m is a real number, then the Hörmander symbol class $S^m = S^m(\mathbf{R}^d \times \mathbf{R}^d)$ consists of smooth functions a such that, for all multi-indices α, β we have

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m - |\alpha|}, \quad x, \xi \in \mathbf{R}^d.$$

Symbol $a \in S^m$ (of order m) defines a pseudo-differential operator $a(x, D) \in \text{Op } S^m$ by

$$(4.1) \quad a(x, D)f(x) = (2\pi)^{-d/2} \int e^{ix \cdot \xi} a(x, \xi) \widehat{f}(\xi) d\xi, \quad f \in C_0^\infty(\mathbf{R}^d)$$

and $a \in S^{-\infty}$ if $a \in \bigcap_{m \in \mathbf{R}} S^m$.

We use the notation $E_s(D) := (1 - \Delta)^s$, for the operator with the symbol $(1 + |\xi|^2)^s$, $s \in \mathbf{R}$. The following result will be used in the proof of Theorem 1.

PROPOSITION 4. *Assume that $q \in [1, \infty]$ and $\chi \in C_0^\infty(\mathbf{R}^d)$.*

(1) *If $a \in S^0$ then the mapping $\chi(x)a(x, D) : \mathcal{F}L^q(\mathbf{R}^d) \rightarrow \mathcal{F}L^q(\mathbf{R}^d)$ is continuous. In particular, if $a(x, \xi) = a(\xi) \in S^0$, then the mapping $a(D) : \mathcal{F}L^q(\mathbf{R}^d) \rightarrow \mathcal{F}L^q(\mathbf{R}^d)$ is continuous.*

(2) *If $a \in S^m$ and $s \in \mathbf{R}$ then $\chi(x)a(x, D) : \mathcal{F}L_s^q(\mathbf{R}^d) \rightarrow \mathcal{F}L_{s-m}^q(\mathbf{R}^d)$. In particular, if $a(x, \xi) = a(\xi) \in S^m$, $a(D) : \mathcal{F}L_s^q(\mathbf{R}^d) \rightarrow \mathcal{F}L_{s-m}^q(\mathbf{R}^d)$.*

Proof. We will prove assertions for $1 \leq q < \infty$. Obvious modification gives the proof for $q = \infty$.

(1) Let there be given $a \in S^0$ and $\chi \in C_0^\infty(\mathbf{R}^d)$. We denote the support of χ by K . The oscillatory integral (4.1) is well defined for the symbol χa and $f \in \mathcal{F}L^q(\mathbf{R}^d)$. Namely, after μ times integration by parts we obtain

$$\chi(x)a(x, D)f(x) = (2\pi)^{-d} \int e^{ix \cdot \theta} \frac{E_\mu(D_x)\chi(x)a(x, \theta)}{(1 + |\theta|^2)^\mu} \widehat{f}(\theta) d\theta,$$

which, by the Hölder inequality, gives

$$|\chi(x)a(x, D)f(x)| \leq (2\pi)^{-d} \left\| \frac{E_\mu(D_x)\chi(x)a(x, \theta)}{\langle \theta \rangle^{2\mu}} \right\|_{L^{q'}} \|\widehat{f}(\theta)\|_{L^q} \leq C \|\widehat{f}\|_{L^q}$$

for $2\mu > \frac{d}{q'}$.

Denote the Fourier transform of $a_\chi(x, \theta) = \chi(x)a(x, \theta)$ with respect to the x variable by $\mathcal{F}_1 a_\chi(\eta, \theta)$. Then, for arbitrary $N \in \mathbf{N}$, there exists $C_N > 0$ such that

$$(4.2) \quad |\mathcal{F}_1 a_\chi(\eta, \theta)| \leq C_N \langle \eta \rangle^{-N}, \quad \theta \in \mathbf{R}^d.$$

Now,

$$\begin{aligned}
& |\mathcal{F}(\chi(x)a(x, D)f(x))(\xi)| \\
&= (2\pi)^{-d} \left| \int_K e^{-ix\xi} \left(\int_{\mathbf{R}^d} e^{ix\theta} \frac{E_\mu(D_x)a_\chi(x, \theta)}{\langle \theta \rangle^{2\mu}} \widehat{f}(\theta) d\theta \right) dx \right| \\
&= (2\pi)^{-d} \left| \int_{\mathbf{R}^d} \frac{\widehat{f}(\theta)}{\langle \theta \rangle^{2\mu}} \left(\int_K e^{-ix(\xi-\theta)} E_\mu(D_x)a_\chi(x, \theta) dx \right) d\theta \right| \\
&= (2\pi)^{-d} \left| \int_{\mathbf{R}^d} \frac{\widehat{f}(\theta)}{\langle \theta \rangle^{2\mu}} (1 + |\xi - \theta|^2)^\mu \mathcal{F}_1 a_\chi(\xi - \theta, \theta) d\theta \right|
\end{aligned}$$

After a change of variables ($\tilde{\theta} = \xi - \theta$) and for $N > \mu$ we obtain

$$\begin{aligned}
& \|\mathcal{F}(\chi(x)a(x, D)f(x))(\xi)\|_{L^q} \\
&= (2\pi)^{-d} \left(\int_{\mathbf{R}^d} \left| \int_{\mathbf{R}^d} \frac{\widehat{f}(\xi - \tilde{\theta})(1 + |\tilde{\theta}|^2)^\mu \mathcal{F}_1 a_\chi(\tilde{\theta}, \xi - \tilde{\theta})}{\langle \xi - \tilde{\theta} \rangle^{2\mu}} d\tilde{\theta} \right|^q d\xi \right)^{1/q} \\
&\leq (2\pi)^{-d} \int_{\mathbf{R}^d} (1 + |\tilde{\theta}|^2)^\mu \left(\int_{\mathbf{R}^d} |\widehat{f}(\xi - \tilde{\theta}) \mathcal{F}_1 a_\chi(\tilde{\theta}, \xi - \tilde{\theta})|^q d\xi \right)^{1/q} d\tilde{\theta} \\
&\leq (2\pi)^{-d} C_N \int_{\mathbf{R}^d} (1 + |\tilde{\theta}|^2)^{\mu-N} \left(\int_{\mathbf{R}^d} |\widehat{f}(\xi - \tilde{\theta})|^q d\xi \right)^{1/q} d\tilde{\theta} \\
&\leq C \|\widehat{f}\|_{L^q},
\end{aligned}$$

where we have used the Minkovski inequality and (4.2).

In particular, if the symbol is a Fourier multiplier $a = a(\xi) \in S^0$ then

$$\|\mathcal{F}(a(D)f)\|_{L^q} \leq C \|\mathcal{F}f\|_{L^q}.$$

(2) We have

$$E_{s-m}(D)\chi(x)a(x, D)f = b(x, D)E_s(D)f,$$

where

$$b(x, D) = E_{s-m}(D)\chi(x)a(x, D)E_{-s}(D).$$

By (1) it follows that the operator $b(x, D)$ is $\mathcal{F}L^q(\mathbf{R}^d)$ continuous which implies $\chi(x)a(x, D)f \in \mathcal{F}L_{s-m}^q(\mathbf{R}^d)$ as claimed. In particular, if $a(x, \xi) = a(\xi) \in S^m$, $a(D) : \mathcal{F}L_s^q(\mathbf{R}^d) \rightarrow \mathcal{F}L_{s-m}^q(\mathbf{R}^d)$. \square

REMARK 1. It is known that the operator $a(x, D)$ whose symbol lies in the Hörmander class S^0 is continuous from L^q to L^q , $1 < q < \infty$, see [9]. In order to prove the continuity in $\mathcal{F}L^q(\mathbf{R}^d)$ it is not sufficient to assume the

boundedness of the corresponding symbol a with respect to the x variable. For such symbol, if $f \in \mathcal{F}L^q(\mathbf{R}^d)$, the convolution $\widehat{a} * f$ does not belong to L^q in general. For that reason we observe the operators of the form $\chi(x)a(x, D)$. Alternatively, we could impose a decay condition on a with respect to the x variable. For example, one can prove that $a(x, D) : \mathcal{F}L^q(\mathbf{R}^d) \rightarrow \mathcal{F}L^q(\mathbf{R}^d)$ is continuous if the symbol a satisfies

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m - |\alpha|} (1 + |x|)^{k - |\beta|},$$

for some $k \in \mathbf{R}$. More details on this topic can be found in [5, 6].

Recall, $a(x, D) \in \text{Op } S^m$ is *elliptic* if there exist $c, C > 0$ such that

$$(4.3) \quad |a(x, \xi)| > c|\xi|^m, \quad \text{if } |\xi| > C,$$

where $a \in S^m$ is the symbol of the operator $a(x, D)$. One says that a is *noncharacteristic* at $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ if (4.3) is valid at infinity in a conic neighborhood of (x_0, ξ_0) . Then one can find $b \in S^{-m}$ such that $a(x, D)b(x, D) - Id$ and $b(x, D)a(x, D) - Id$ belong to $\text{Op } S^{-\infty}$ in a conic neighborhood of (x_0, ξ_0) . The complement of the set of noncharacteristic points of $a(x, D)$ is denoted by *char* $a(x, D)$.

PROPOSITION 5. *Let $a \in S^m$ be elliptic and assume that $f \in \mathcal{F}L_{t, \text{loc}}^q(\mathbf{R}^d)$ for some $q \in [1, \infty]$ and for some $t \in \mathbf{R}$. If $a(x, D)f \in \mathcal{F}L_{s, \text{loc}}^q(\mathbf{R}^d)$ then $f \in \mathcal{F}L_{s+m, \text{loc}}^q(\mathbf{R}^d)$ and, for every $\chi \in C_0^\infty(\mathbf{R}^d)$, we have*

$$(4.4) \quad \|\chi f\|_{\mathcal{F}L_{s+m}^q} \leq C_{s, t} (\|\chi a(x, D)f\|_{\mathcal{F}L_s^q} + \|\chi f\|_{\mathcal{F}L_t^q}).$$

In particular, if $a(x, D) = a(D) \in S^m$ is elliptic, then (4.4) holds without χ .

Proof. The ellipticity condition implies that there is an operator $b(x, D) \in \text{Op } S^{-m}$ such that

$$\chi f = b(x, D)a(x, D)\chi f + r(x, D)\chi f,$$

for some $r \in S^{-\infty}$. Hence $b(x, D)$ is continuous from $\mathcal{F}L_s^q(\mathbf{R}^d)$ to $\mathcal{F}L_{s+m}^q(\mathbf{R}^d)$ and $r(x, D)$ is continuous from $\mathcal{F}L_t^q(\mathbf{R}^d)$ to $\mathcal{F}L_{s+m}^q(\mathbf{R}^d)$. This implies

$$\begin{aligned} \|\chi f\|_{\mathcal{F}L_{s+m}^q} &\leq \|b(x, D)a(x, D)\chi f\|_{\mathcal{F}L_{s+m}^q} + \|r(x, D)\chi f\|_{\mathcal{F}L_{s+m}^q} \\ &\leq C_{s, t} (\|\chi a(x, D)f\|_{\mathcal{F}L_s^q} + \|\chi f\|_{\mathcal{F}L_t^q}). \end{aligned}$$

□

REMARK 2. The above proposition can be reformulated in the language of the symbol class $S_{\text{loc}}^m(X \times \mathbf{R}^d)$, where X is an open set in \mathbf{R}^d . This class is introduced in [2] as the starting point in the study of pseudo-differential operators on manifolds.

5. Wave front sets and operators

In this section we observe symbols introduced in [2] in order to study pseudo-differential operators on manifolds. We say that a continuous linear map $A : C_0^\infty(X) \rightarrow C^\infty(X)$ is a pseudo-differential operator of order m in X , $A \in \Psi^m(X)$, if for arbitrary $\phi, \psi \in C_0^\infty(X)$ the operator $f \mapsto \phi A(\psi f)$ is in $\text{Op } S^m$. For example, the restriction of $a(x, D) \in \text{Op } S^m$ to X belongs to $\Psi^m(X)$.

By the Schwartz kernel theorem, for any $A \in \Psi^m(X)$ there exists $K(x, y) \in C^\infty(X \times X \setminus \Delta_X)$, where Δ_X denotes the diagonal $\{(x, x) ; x \in X\}$, such that $Au(x) = \int K(x, y)u(y)dy$. The pseudo-differential operator with the kernel K is *properly supported* if

$$\text{supp}K \ni (x, y) \mapsto x \in X \quad \text{and} \quad \text{supp}K \ni (x, y) \mapsto y \in X$$

are proper, that is, the inverse of any compact set in X is a compact set in $\text{supp}K \subset X \times X$. According to [2, Proposition 18.1.22], every $A \in \Psi^m(X)$ can be decomposed as $A = A_0 + A_1$ where $A_1 \in \Psi^m(X)$ is properly supported and the kernel of A_0 is in C^∞ . In that sense it is no essential restriction to require proper supports in the following statements.

THEOREM 1. *Assume that $q \in [1, \infty]$, $s \in \mathbf{R}$, $f \in \mathcal{D}'(\mathbf{R}^d)$ and $(x_0, \xi_0) \in \mathbf{R}^d \times \mathbf{R}^d \setminus 0$. Then the following conditions are equivalent.*

- (1) $(x_0, \xi_0) \notin WF_{\mathcal{F}L_s^q}(f)$
- (2) $(x_0, \xi_0) \notin WF_{\mathcal{F}L^q}(Af)$ for some properly supported $A \in \Psi^s(X)$ which is noncharacteristic at (x_0, ξ_0) .
- (3) There is a conic neighborhood $U \times \Gamma_0$ of (x_0, ξ_0) in $X \times \mathbf{R}^d \setminus 0$ such that Bf in $\mathcal{F}L_{s-m, \text{loc}}^q(X)$ for every properly supported pseudo-differential operator $B \in \Psi^m(X)$ with the symbol of class $-\infty$ outside $U \times \Gamma_0$.

Proof. We follow the proof of Theorem 8.4.8 [3] which concerns the classical wave front set.

Assume that (1) holds with $\omega_0(\xi) = \langle \xi \rangle^s$. Then (3.5) holds for some conic neighborhood Γ of ξ_0 and for some $\varphi \in C_0^\infty(X)$ such that $\varphi(x_0) \neq 0$. Let $q(\xi) \in C^\infty$ be a homogeneous function of degree s for $|\xi| \geq 1$, with support in Γ . We define $A = \varphi q(D)\varphi$, where φ is from (3.5). Then $A \in \Psi^s(X)$ and (by (3.5))

$$\|Af\|_{\mathcal{F}L^q} \leq C\|q(D)\varphi f\|_{\mathcal{F}L^q} < \infty.$$

Moreover, the symbol of A is $q(\xi) \pmod{S^{-\infty}}$ near x_0 , which proves (2).

Now, assume that (3) holds. To prove (1) it is sufficient to find $\varphi \in C_0^\infty$ such that $\text{supp } \varphi$ is sufficiently close to x and a conic neighborhood Γ of ξ_0 such that (3.5) holds. By (3), for given $B \in \Psi^m(X)$ we may choose $\phi, \psi \in C_0^\infty(X)$ and $q(\xi) \in C^\infty$ a homogeneous of degree m for $|\xi| \geq 1$, with $\psi = 1$ in a neighborhood of $\text{supp } \phi$ and $\text{supp } \phi \times \text{supp } q \subset U \times \Gamma_0$, so that $B = \psi q(D)\phi$.

Observe

$$\mathcal{F}(q(D)\phi f) = \mathcal{F}(\psi q(D)\phi f) + \mathcal{F}((1 - \psi)q(D)\phi f).$$

By (3) it follows that $\psi q(D)\phi f \in \mathcal{F}L_{s-m}^q$ and $(1 - \psi)q(D)\phi$ is of order $-\infty$. Therefore (3.5) holds with $\omega_0(\xi) = \langle \xi \rangle^s$, $\Gamma = \Gamma_0$ and $\varphi = q(D)\phi$.

Finally, assume that (2) holds and choose a closed conic neighborhood $U \times \Gamma_0$ of (x_0, ξ_0) such that the symbol $a(x, \xi)$ of A satisfies $|a(x, \xi)| > c|\xi|^m$, if $\xi \in \Gamma_0$, $|\xi| > C$, for some constants $c, C > 0$. Therefore, for every B as in condition (3), we can find a properly supported $\tilde{B} \in \Psi^{m-s}(X)$ such that

$$B - \tilde{B}A \in \Psi^{-\infty}(X).$$

Hence $Bf - \tilde{B}Af \in C^\infty(X)$. By the assumption and Proposition 4 (2) it follows that $\tilde{B}Af \in \mathcal{F}L_{s-m,loc}^q(X)$ and therefore $Bf \in \mathcal{F}L_{s-m,loc}^q(X)$ which completes the proof. \square

COROLLARY 1. *Assume that $q \in [1, \infty]$, $s \in \mathbf{R}$, $f \in \mathcal{D}'(\mathbf{R}^d)$ and $(x_0, \xi_0) \in \mathbf{R}^d \times \mathbf{R}^d \setminus 0$.*

- (1) *If $(x_0, \xi_0) \notin WF_{\mathcal{F}L_s^q}(f)$ then $(x_0, \xi_0) \notin WF_{\mathcal{F}L_{s-m}^q}(Af)$ for every properly supported $A \in \Psi^m(X)$.*
- (2) *If $(x_0, \xi_0) \notin WF_{\mathcal{F}L_{s-m}^q}(Af)$ for some properly supported $A \in \Psi^m(X)$ which is noncharacteristic at (x_0, ξ_0) , then $(x_0, \xi_0) \notin WF_{\mathcal{F}L_s^q}(f)$.*
- (3) *There is a neighborhood U of x_0 such that $(x, \xi_0) \notin WF_{\mathcal{F}L_s^{q,r}}(f)$ for every $x \in U$ and for every $s \in \mathbf{R}$ if and only if $(x_0, \xi_0) \notin WF(f)$.*

Proof. We use the same idea as in the proof of [3, Theorem 8.4.8]. Here, we present only the proof of (3) and remark that an equivalent statement is given in [5] via the so called superposition type wave-front sets.

By the assumption, for every $\chi \in C_0^\infty(U)$, $\chi(x_0) \neq 0$, and for every s , $|\chi f|_{\mathcal{F}L_s^{q,r}} < \infty$ for some open cone $\Gamma = \Gamma_{\xi_0}$. Choose $\varphi \in C_0^\infty(U)$ and $q(\xi)$ homogeneous of degree 0 for $|\xi| > 1$, so that

$$\varphi(x_0) = q(\xi_0/|\xi_0|) = 1, \quad \text{supp } \varphi \times \text{supp } q \subset U \times \Gamma_0.$$

Then $q(D)\varphi(x)f \in \mathcal{F}L_{s,loc}^q$ for every s by condition (3) in Theorem 1, for we can cover $\text{supp } \varphi \times \text{supp } q$ by a finite number of conic neighborhoods with the properties required there and decompose the symbol correspondingly. Hence $q(D)\varphi(x)f \in C^\infty$.

Since $q(\xi)$ is homogeneous of degree 0 for $|\xi| > 1$ and the kernel of $q(D)$ is given by $K(x - y) = (2\pi)^{-d} \int_{\mathbf{R}^d} e^{(x-y)\xi} q(\xi) d\xi$ (in the distributional sense), we conclude that $K \in C^\infty$ except at 0 and it is rapidly decreasing. It follows that $q(D)\varphi(x)f \in \mathcal{S}(\mathbf{R}^d)$, hence $q(\xi)\mathcal{F}(\varphi f)(\xi)$ is rapidly decreasing and therefore $(x_0, \xi_0) \notin WF(f)$.

On the other hand, if $(x_0, \xi_0) \notin WF(f)$ then, by the definition, it immediately follows that there is a neighborhood U of x_0 such that $(x, \xi_0) \notin WF_{\mathcal{F}L_s^q, r}(f)$ for every $x \in U$ and for every $s \in \mathbf{R}$. The proof is complete. \square

THEOREM 2. *Let $f \in \mathcal{S}'(\mathbf{R}^d)$ and let $A \in \Psi^m(X)$ be properly supported. Then we have the microlocal property*

$$WF_{\mathcal{F}L_{s-m}^q}(Af) \subset WF_{\mathcal{F}L_s^q}(f) \subset WF_{\mathcal{F}L_{s-m}^q}(Af) \cup \text{char}A,$$

where $\text{char}A$ denotes the set of characteristic points of A .

Proof. The statement follows directly from Corollary 1 (1) and (2). \square

6. Modulation spaces

Assume that $\varphi \in \mathcal{S}'(\mathbf{R}^d)$ is fixed. Then the short-time Fourier transform of $f \in \mathcal{S}'(\mathbf{R}^d)$ with respect to φ is defined by

$$(V_\varphi f)(x, \xi) = \mathcal{F}(f \cdot \overline{\varphi(\cdot - x)})(\xi).$$

We note that the left-hand side makes sense, since it is the partial Fourier transform of the tempered distribution

$$F(x, y) = (f \otimes \overline{\varphi})(y, y - x)$$

with respect to the y -variable.

We usually assume that $\varphi \in \mathcal{S}(\mathbf{R}^d)$, and in this case $V_\varphi f$ takes the form

$$(6.1) \quad V_\varphi f(x, \xi) = (2\pi)^{-d/2} \int f(y) \overline{\varphi(y - x)} e^{-iy\xi} dy.$$

Assume that $\omega \in \mathcal{P}(\mathbf{R}^{2d})$, $p, q \in [1, \infty]$, and that $\varphi \in \mathcal{S}(\mathbf{R}^d) \setminus 0$. Then the modulation space $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbf{R}^d)$ such that

$$(6.2) \quad \begin{aligned} \|f\|_{M_{(\omega)}^{p,q}} &= \|f\|_{M_{(\omega)}^{p,q,\varphi}} \\ &\equiv \left(\int \left(\int |V_\varphi f(x, \xi) \omega(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} < \infty \end{aligned}$$

(with obvious interpretation when $p = \infty$ or $q = \infty$).

If $\omega = 1$, then the notation $M^{p,q}(\mathbf{R}^d)$ is used instead of $M_{(\omega)}^{p,q}(\mathbf{R}^d)$. Moreover we set $M_{(\omega)}^p(\mathbf{R}^d) = M_{(\omega)}^{p,p}(\mathbf{R}^d)$ and $M^p(\mathbf{R}^d) = M^{p,p}(\mathbf{R}^d)$.

Locally, the spaces $\mathcal{F}L_{(\omega)}^q(\mathbf{R}^d)$ and $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ coincide, in the sense that

$$\mathcal{F}L_{(\omega)}^q(\mathbf{R}^d) \cap \mathcal{E}'(\mathbf{R}^d) = M_{(\omega)}^{p,q}(\mathbf{R}^d) \cap \mathcal{E}'(\mathbf{R}^d)$$

(see Theorem 2.1 and Remark 4.4 in [7]). This result is extended in [5] in the context of wave front sets.

In this section we define wave-front sets with respect to modulation spaces, and claim that they coincide with wave-front sets of Fourier Lebesgue types. In particular, any property valid for wave-front set of Fourier Lebesgue type carry over to wave-front set of modulation space type.

Assume that $\varphi \in \mathcal{S}(\mathbf{R}^d) \setminus 0$, $\omega \in \mathcal{P}(\mathbf{R}^{2d})$, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ is an open cone and $p, q \in [1, \infty]$ are fixed. For any $f \in \mathcal{S}'(\mathbf{R}^d)$, let

$$(6.3) \quad \begin{aligned} |f|_{M_{(\omega)}^{p,q,\Gamma}} &= |f|_{M_{(\omega)}^{p,q,\Gamma,\varphi}} \\ &\equiv \left(\int_{\Gamma} \left(\int_{\mathbf{R}^d} |V_{\varphi} f(x, \xi) \omega(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} \end{aligned}$$

(with obvious interpretation when $p = \infty$ or $q = \infty$). We note that $|\cdot|_{M_{(\omega)}^{p,q,\Gamma}}$ defines a semi-norms on \mathcal{S}' which might attain the value $+\infty$. If $\Gamma = \mathbf{R}^d \setminus 0$, then $|f|_{M_{(\omega)}^{p,q,\Gamma}}$ agrees with the modulation space norm $\|f\|_{M_{(\omega)}^{p,q}}$ of f .

We let $\Theta_{M_{(\omega)}^{p,q}}(f) = \Theta_{M_{(\omega)}^{p,q,\varphi}}(f)$ be the sets of all $\xi \in \mathbf{R}^d \setminus 0$ such that $|f|_{M_{(\omega)}^{p,q,\Gamma,\varphi}} < \infty$, for some $\Gamma = \Gamma_{\xi}$. We also let $\Sigma_{M_{(\omega)}^{p,q,\varphi}}(f)$ be the complement of $\Theta_{M_{(\omega)}^{p,q,\varphi}}(f)$ in $\mathbf{R}^d \setminus 0$. Then $\Theta_{M_{(\omega)}^{p,q,\varphi}}(f)$ and $\Sigma_{M_{(\omega)}^{p,q,\varphi}}(f)$ are open respectively closed subsets in $\mathbf{R}^d \setminus 0$. We have now the following continuation of Proposition 1. The proof is given in [5].

THEOREM 3. *Assume that $p, q \in [1, \infty]$, $\varphi \in C_0^{\infty}(\mathbf{R}^d) \setminus 0$, $\chi \in C^{\infty}(\mathbf{R}^d)$, and that $\omega \in \mathcal{P}(\mathbf{R}^{2d})$. Also assume that $f \in \mathcal{E}'(\mathbf{R}^d)$. Then*

$$(6.4) \quad \Theta_{M_{(\omega)}^{p,q,\varphi}}(f) = \Theta_{\mathcal{FL}_{(\omega)}^q}(f), \quad \Sigma_{M_{(\omega)}^{p,q,\varphi}}(f) = \Sigma_{\mathcal{FL}_{(\omega)}^q}(f),$$

and

$$(6.5) \quad \Sigma_{M_{(\omega)}^{p,q,\varphi}}(\chi f) \subseteq \Sigma_{M_{(\omega)}^{p,q,\varphi}}(f), \quad \Sigma_{\mathcal{FL}_{(\omega)}^q}(\chi f) \subseteq \Sigma_{\mathcal{FL}_{(\omega)}^q}(f).$$

COROLLARY 2. *Assume that $p, q \in [1, \infty]$, $f \in \mathcal{D}'(\mathbf{R}^d)$, $\varphi \in C_0^{\infty}(\mathbf{R}^d) \setminus 0$, $x_0, y_0 \in \mathbf{R}^d$, $\xi_0 \in \mathbf{R}^d \setminus 0$ and that $\omega \in \mathcal{P}(\mathbf{R}^{2d})$. Also let $\omega_0(\xi) = \omega(y_0, \xi)$. Then the following conditions are equivalent:*

1. *there exists an open cone $\Gamma = \Gamma_{\xi_0}$ and $\chi \in C_0^{\infty}(\mathbf{R}^d)$ such that $\chi(x_0) \neq 0$, and $|\chi f|_{M_{(\omega)}^{p,q,\Gamma,\varphi}} < \infty$ (i. e. $\xi_0 \in \Theta_{M_{(\omega)}^{p,q}}(\chi f)$);*
2. *there exists an open cone $\Gamma = \Gamma_{\xi_0}$ and $\chi \in C_0^{\infty}(\mathbf{R}^d)$ such that $\chi(x_0) \neq 0$, and $|\chi f|_{\mathcal{FL}_{(\omega)}^q} < \infty$ (i. e. $\xi_0 \in \Theta_{\mathcal{FL}_{(\omega)}^q}(\chi f)$);*
3. *there exists an open cone $\Gamma = \Gamma_{\xi_0}$ and $\chi \in C_0^{\infty}(\mathbf{R}^d)$ such that $\chi(x_0) \neq 0$, and $|\chi f|_{\mathcal{FL}_{(\omega_0)}^q} < \infty$ (i. e. $\xi_0 \in \Theta_{\mathcal{FL}_{(\omega_0)}^q}(\chi f)$).*

The following definition makes sense in view of Corollary 2.

DEFINITION 3. Assume that $p, q \in [1, \infty]$, $f \in \mathcal{D}'(\mathbf{R}^d)$ and $\omega \in \mathcal{P}(\mathbf{R}^{2d})$. The wave-front set $WF_{M_{(\omega)}^{p,q}}(f)$ with respect to $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ consists of all pairs (x_0, ξ_0) in $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_{M_{(\omega)}^{p,q,\varphi}}(\chi f)$;

By Corollary 2 it follows that

$$WF_{M_{(\omega_1)}^{p_1,q}}(f) = WF_{M_{(\omega_2)}^{p_2,q}}(f)$$

when $p_1, p_2 \in [1, \infty]$ and

$$C^{-1}\omega_2(x, \xi)\langle x \rangle^{-N} \leq \omega_1(x, \xi) \leq C\omega_2(x, \xi)\langle x \rangle^N,$$

for some constants C and N . By the same corollary it follows that the following holds.

PROPOSITION 6. Assume that $p, q \in [1, \infty]$, $f \in \mathcal{D}'(\mathbf{R}^d)$, $\omega_0 \in \mathcal{P}(\mathbf{R}^d)$ and $\omega \in \mathcal{P}(\mathbf{R}^{2d})$ are such that $\omega_0(\xi) = \omega(y_0, \xi)$ for some $y_0 \in \mathbf{R}^d$. Then

$$WF_{\mathcal{F}L_{(\omega_0)}^q}(f) = WF_{\mathcal{F}L_{(\omega)}^q}(f) = WF_{M_{(\omega)}^{p,q}}(f).$$

We also note that if $f \in \mathcal{E}'(\mathbf{R}^d)$, then it follows from Corollary 2 that

$$f \in \mathcal{F}L_{(\omega_0)}^q \iff f \in M_{(\omega)}^{p,q} \iff WF_{\mathcal{F}L_{(\omega_0)}^q}(f) = WF_{M_{(\omega)}^{p,q}}(f) = \emptyset.$$

In particular, we recover Theorem 2.1 and Remark 4.4 in [7].

References

- [1] E. CORDERO, F. NICOLA, L. RODINO, *Boundedness of Fourier integral operators on $\mathcal{F}L^p$ spaces*, preprint, 2008, available at arXiv:0801.1444v2
- [2] L. HÖRMANDER *The Analysis of Linear Partial Differential Operators, vol III*, Springer-Verlag, Berlin Heidelberg NewYork Tokyo, 1994.
- [3] L. HÖRMANDER, *Lectures on Nonlinear Hyperbolic Differential Equations*, Springer-Verlag, Berlin, 1997.
- [4] K. A. OKOUDJOU, *A Beurling-Helson type theorem for modulation spaces*, preprint, 2008, a available at arXiv:0801.1338
- [5] S. PILIPOVIĆ, N. TEOFANOV, J. TOFT, *Micro-local analysis in Fourier Lebesgue and modulation spaces. Part I*, preprint, 2008, a version is available at arXiv:0804.1730

- [6] S. PILIPOVIĆ, N. TEOFANOV, J. TOFT, *Micro-local analysis in Fourier Lebesgue and modulation spaces. Part II, preprint, 2008, a version is available at arXiv:0805.4476*
- [7] M. RUZHANSKY, M. SUGIMOTO, N. TOMITA, J. TOFT, *Changes of variables in modulation and Wiener amalgam spaces, Preprint, 2008, Available at arXiv:0803.3485v1.*
- [8] T. STROHMER, *pseudo-differential operators and Banach algebras in mobile communications, Appl. Comput. Harmon. Anal.* **20** (2006), 237–249.
- [9] M. W. WONG, *An Introduction To Pseudo-differential Operators 2nd Edition, World Scientific, 1999.*

AMS Subject Classification: ?????, ?????.

Stevan PILIPOVIĆ, Nenad TEOFANOV
Department of Mathematics and Informatics
University of Novi Sad
Trg D. Obradovića 4
21000 Novi Sad, SERBIA
e-mail: email@provider

Joachim TOFT
Department of Mathematics and Systems Engineering
Växjö University
Växjö, SWEDEN
e-mail: joachim.toft@vxu.se