SIMPLICITY IN HOMOGENEOUS GRAPHS AND DIGRAPHS

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Abstract. An interval of a relational structure defined on a set A is a subset $I \subseteq A$ whose elements are related to each point in $A \setminus I$ in the same way; a structure having no nontrivial intervals is simple. In this note we characterise all simple homogeneous graphs and digraphs in terms of their finite subgraphs. Furthermore, we characterise a broader class of semisimple homogeneous graphs and digraphs, in which every nontrivial interval induces a subgraph with no edges.

1. Introduction

Let $\mathcal{L}$ be a relational first-order language. An $\mathcal{L}$-*structure* is a relational structure $\mathcal{A} = (A, R^{\mathcal{A}})_{R\in\mathcal{L}}$, where each symbol from $\mathcal{L}$ is interpreted in $\mathcal{A}$ by a relation on A of appropriate arity. An *interval* in $\mathcal{A}$ is a subset $\varnothing \neq I \subseteq A$ with the property that every element $a \in A \setminus I$ relates to each $x \in I$ 'in the same way'; more precisely, for every $R \in \mathcal{L}$, $\mathrm{ar}(R) = n$, and all tuples $\mathbf{x}, \mathbf{y} \in A^n \setminus I^n$ such that for all i either $x_i = y_i$ or $x_i, y_i \in I$, we have

$$\mathbf{x} \in R^{\mathcal{A}} \quad \text{if and only if} \quad \mathbf{y} \in R^{\mathcal{A}}.$$

(This general notion clearly stems from the familiar notion of an interval in a linearly ordered set $(A, <)$: indeed, any point x not belonging to an interval I is either 'above' or 'below' all the elements from I.) Of course, any singleton set is an interval, as is the empty set and the whole of A; these intervals are called *trivial*, while all the others (if any) are *proper*. A structure is *simple* if it has no proper intervals.

Concerning simple relational structures, a considerable amount of attention was devoted to the question of embedding an arbitrary structure into a simple one: see, for example, [5, 6] for the particular case of tournaments. A broader approach to this question was taken in the recent article by Brignall, Ruškuc and Vatter [1], where various types of (finite) combinatorial structures were covered. This note is inspired, albeit not directly motivated, by the latter contribution, where in the closing passages the authors hint towards studying the notion of simplicity in infinite structures as well. Here we supply transparent characterisations of *homogeneous* graphs and digraphs (of arbitrary cardinality) that are simple in the sense just described. Recall that a structure $\mathcal{A}$ is said to be homogeneous (the term *ultrahomogeneous* also appears frequently in the model-theoretical literature) if any isomorphism $\mathcal{B} \to \mathcal{B}'$ between its finitely generated substructures extends to an automorphism of $\mathcal{A}$. (In relational structures 'finitely generated' refers simply to finite induced substructures.) For example, countably infinite homogeneous graphs

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were described by Lachlan and Woodrow [12]; the same task was accomplished for posets by Schmerl [13] and for directed graphs in the memoir of Cherlin [4].

Loosely speaking, homogeneous structures possess an extraordinary amount of symmetry; therefore, at first it appears as quite difficult for such structures not to be simple. However, let us not forget about the example of $\mathbb{Q}$, the linear order of rationals, which is one of the epitomes of a homogeneous structure [8] and still has many proper intervals. By a fundamental result of Fraïssé [8, 10], a countable homogeneous relational structure is uniquely determined by its *age*, the set of isomorphism types of its finite (induced) substructures; keeping this in mind, our description of simple homogeneous graphs and digraphs will be given in terms of presence of certain finite subgraphs in their ages. As it turns out, rather small subgraphs suffice to decide the considered simplicity question. Related to our principal goal, a natural intermediate notion of *semisimple* graphs and digraphs arises: namely, it describes (di)graphs in which all proper intervals contain no edges (we call such intervals *null*). In this note we also characterise semisimple homogenous graphs and digraphs.

At this point, we would like to stress that simple and semisimple *countable* homogeneous graphs and digraphs can easily be obtained by inspecting the exhaustive lists of such objects obtained by Lachlan and Woodrow in [12] and Cherlin in [4], respectively. In this paper, however, we describe simple and semisimple homogeneous (di)graphs of *arbitrary* cardinalities. Hence, this paper can be thought of as a contribution towards the goal of classifying uncountable homogeneous (di)graphs, where not much is known.

2. Preliminaries

Just to fix the terminology (which is standard, yet slightly different from [1]), by a *directed graph* (or a *digraph*) we mean an ordered pair $\mathcal{D} = (V, E)$ where $E \subseteq V \times V$ such that

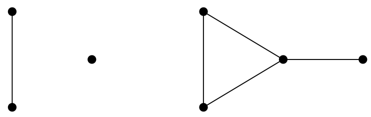
(1) if $(x, y) \in E$ and $x \neq y$ then $(y, x) \notin E$, and
(2) $(x, x) \notin E$ for all $x \in V$.

Usually, we write $x \to y$ for $(x, y) \in E(\mathcal{D})$. If $X, Y \subseteq V(\mathcal{D})$ are nonempty then $X \rightrightarrows Y$ stands for $x \to y$ for all $x \in X$ and all $y \in Y$. Instead of $\{x\} \rightrightarrows Y$ and $X \rightrightarrows \{y\}$ we write $x \rightrightarrows Y$ and $X \rightrightarrows y$, respectively. We denote by $x \sim y$ the fact that $x \to y$ or $y \to x$ and write $x \not\sim y$ if it is not the case that $x \sim y$. The same notation is utilised for *undirected* graphs: $x \sim y$ signifies that there is an edge between the vertices x and y, while otherwise $x \not\sim y$. By analogy with the previous definitions, $x \sim Y$ means that x is adjacent to each vertex from the set Y.

For a (di)graph $\mathcal{G}$ we specify some further notation. For $\varnothing \neq W \subseteq V(\mathcal{G})$ by $\mathcal{G}[W]$ we denote the *sub(di)graph of* $\mathcal{G}$ *induced by* W. The fact that $\mathcal{G}'$ is isomorphic to an (induced) sub(di)graph of $\mathcal{G}$ is written $\mathcal{G}' \leqslant \mathcal{G}$. If $\mathcal{G}$ is a digraph, then for $A, B \subseteq V(\mathcal{G})$ we let $E_{\mathcal{G}}(A, B) = \{(a, b) \in E(\mathcal{G}) : a \in A, b \in B\}$; for the undirected case, ordered pairs are replaced by unordered ones.

A digraph $\mathcal{D}$ is *disconnected* if it is not weakly connected. A *connected component* of $\mathcal{D}$ is a maximal set of vertices $S \subseteq V(\mathcal{D})$ such that $\mathcal{D}[S]$ is weakly connected.

We say that a digraph $\mathcal{D}$ *has a cycle* if there exist an $n \geqslant 3$ and $x_1, \ldots, x_n \in V(\mathcal{D})$ such that $x_1 \to x_2 \to \ldots \to x_n \to x_1$. On the other hand, $\mathcal{D}$ is said to be *acyclic* if it does not have a cycle.



$\mathcal{Z}_3'$ $\mathcal{X}_4$

FIGURE 1. Graphs $\mathcal{Z}_3'$ and $\mathcal{X}_4$

A digraph $\mathcal{D}$ is *transitive* if $x \to y \to z$ implies $x \to z$ for all $x, y, z \in V(\mathcal{D})$. A *tournament* is a digraph $\mathcal{T}$ such that either $x \to y$ or $y \to x$, whenever x and y are distinct vertices of $\mathcal{T}$.

Let us now take a look at the notion of an interval for the particular cases of graphs and digraphs. Namely, it is easy to see that if $\mathcal{G}$ is an undirected graph and $\varnothing \neq I \subseteq V(\mathcal{G})$ then I is an interval in $\mathcal{G}$ if and only if for all $x \in V(\mathcal{G}) \setminus I$ we have:

- if $x \sim y$ for some $y \in I$ then $x \sim I$;
- if $x \not\sim y$ for some $y \in I$ then $x \not\sim z$ for all $z \in I$.

Similarly, a nonempty set of vertices I in a digraph $\mathcal{D}$ is an interval in $\mathcal{D}$ if and only if the following holds for all $x \in V(D) \setminus I$:

- if $x \to y$ for some $y \in I$ then $x \rightrightarrows I$;
- if $y \to x$ for some $y \in I$ then $I \rightrightarrows x$;
- if $x \not\sim y$ for some $y \in I$ then $x \not\sim z$ for all $z \in I$.

We finish this brief section with an easy observation, explaining why it is sufficient to focus our attention to connected graphs and digraphs only. Namely, if S is a connected component of $\mathcal{G}$ such that $E(\mathcal{G}[S]) \neq \varnothing$ then clearly S is a proper interval in $\mathcal{G}$ which is not null.

Lemma 1. *Let $\mathcal{G}$ be a disconnected (di)graph. Then $\mathcal{G}$ is semisimple if and only if $E(\mathcal{G}) = \varnothing$.*

Therefore, if $\mathcal{G}$ is a disconnected (di)graph, it is almost never semisimple (and thus not simple), since every nontrivial connected component of $\mathcal{G}$ is a proper non-null interval in $\mathcal{D}$.

3. SIMPLE HOMOGENEOUS GRAPHS

As a starter, we describe the homogeneous graphs allowing no proper intervals. To this end, we consider the graphs given in Fig. 1.

Lemma 2. *Let $\mathcal{G}$ be a homogeneous graph. Then $\mathcal{G}$ has no proper null intervals if and only if either $\mathcal{Z}_3' \leqslant \mathcal{G}$, or $\mathcal{G}$ is complete.*

Proof. ($\Rightarrow$): Assume $\mathcal{G}$ is not complete. Then there exist $x, y \in V(\mathcal{G})$ such that $x \not\sim y$. Since $\{x, y\}$ is by assumption not an interval in $\mathcal{G}$, there must be a vertex z adjacent to one of x, y and non-adjacent to the other. Hence, $\mathcal{G}$ has an induced subgraph isomorphic to $\mathcal{Z}_3'$.

($\Leftarrow$): The assertion is clear if $\mathcal{G}$ is complete; therefore, assume that $\mathcal{Z}_3'$ is in the age of $\mathcal{G}$. Let $I \subset V(\mathcal{G})$ be such that $|I| \geqslant 2$ and the subgraph $\mathcal{G}[I]$ contains no edges. By homogeneity of $\mathcal{G}$ there is an automorphism φ of $\mathcal{G}$ that maps a pair of non-adjacent vertices of $\mathcal{Z}_3'$ onto any fixed pair $x, y \in I$, $x \neq y$. By applying φ to the third vertex of $\mathcal{Z}_3'$, we obtain the existence of a vertex $z \in V(\mathcal{G})$ such that $z \sim x$ and $z \not\sim y$, showing that I cannot be an interval in $\mathcal{G}$ (note that $z \notin I$ because of $z \sim x$). □

Lemma 3. *Let $\mathcal{G}$ be a connected homogeneous graph. If $\mathcal{X}_4 \leqslant \mathcal{G}$, then $\mathcal{G}$ is simple.*

Proof. Since $\mathcal{Z}_3' \leqslant \mathcal{X}_4$, the graph $\mathcal{G}$ has no proper null intervals. Therefore, if I is a proper interval $\mathcal{G}$, then $\mathcal{G}[I]$ contains an edge, say $x \sim y$. Furthermore, since $\mathcal{G}$ is connected, there is a vertex z such that $z \sim x_0$ for some $x_0 \in I$; then, however, $z \sim I$, and in particular $z \sim x$ and $z \sim y$. By the homogeneity of $\mathcal{G}$, for any isomorphism between the triangle of $\mathcal{X}_4$ and the triangle $\mathcal{G}[\{x, y, z\}]$ there is an automorphism of $\mathcal{G}$ extending such an isomorphism, implying the existence of a vertex $u \in V(\mathcal{G})$ such that $u \sim x$, while $u \not\sim y$ and $u \not\sim z$. The latter condition shows that $u \notin I$, whence u witnesses that I is not an interval of $\mathcal{G}$. A contradiction. □

The following is the main result of this section.

Theorem 4. *Let $\mathcal{G}$ be a homogeneous graph with at least three vertices. Then $\mathcal{G}$ is simple if and only if $\mathcal{G}$ is connected and $\mathcal{Z}_3' \leqslant \mathcal{G}$.*

Proof. ($\Rightarrow$): If $\mathcal{G}$ is simple and it has at least three vertices, then it obviously cannot be complete. By Lemma 2 it immediately follows that $\mathcal{Z}_3' \leqslant \mathcal{G}$, as $\mathcal{G}$ has no proper intervals and thus, in particular, no null ones. Also, $\mathcal{G}$ must be connected by Lemma 1.

($\Leftarrow$): Seeking a contradiction, assume that $\mathcal{G}$ has a proper interval I; by Lemma 2 it cannot be null, so let $x, y \in I$ be such that $x \sim y$. Now partition the set $V(\mathcal{G}) \setminus I$ into $V^+ = \{z \in V(\mathcal{G}) \setminus I : z \sim I\}$ and $V^- = \{z \in V(\mathcal{G}) \setminus I : z \not\sim I\}$. Since $\mathcal{G}$ is connected, we must have $V^+ \neq \varnothing$. For similar reasons, if $V^- \neq \varnothing$, then there should be an edge $z \sim u$ such that $z \in V^+$ and $u \in V^-$. However, in that case $\mathcal{G}[\{x, y, z, u\}] \cong \mathcal{X}_4$, contradicting (by Lemma 3) the assumption that $\mathcal{G}$ is not simple. Therefore, we conclude that $V^- = \varnothing$, which means that $\mathcal{G} = \mathcal{G}[I] \vee \mathcal{G}[V \setminus I]$, i.e. every vertex from I is adjacent to eery vertex from $V \setminus I$. But $\mathcal{G}$ is assumed to embed $\mathcal{Z}_3'$, which is in the present situation possible only if either $\mathcal{Z}_3' \leqslant \mathcal{G}[I]$, or $\mathcal{Z}_3' \leqslant \mathcal{G}[V \setminus I]$. By taking any vertex from $V \setminus I$ in the former, or from I in the latter case, we again arrive at $\mathcal{X}_4 \leqslant \mathcal{G}$, a contradiction. Hence, $\mathcal{G}$ must be simple. □

Downsizing to the countable case, we recall the complete catalog of *countably infinite* homogeneous graphs that has been given by Lachlan and Woodrow [12]:

(a) $m\mathcal{K}_n$, the disjoint union of m complete graphs on n vertices, where $m, n \leqslant \aleph_0$ and at least one of m, n equals $\aleph_0$;
(b) the complements of (a);
(c) the *Henson graphs* H_n, $n \geqslant 3$, the homogeneous graphs whose ages comprise all finite graphs $\mathcal{G}$ such that $\mathcal{K}_n \not\leqslant \mathcal{G}$ (see [9]);
(d) the complements of (c);
(e) the *random graph* [2, 3], which is the unique countable homogeneous graph whose age contains all finite graphs.

None of the graphs from (a) and (b) are simple; we have that $m\mathcal{K}_n$ is semisimple precisely when $(m, n) = (\aleph_0, 1)$, while $\overline{m\mathcal{K}_n}$ is semisimple if and only if $(m, n) \neq (\aleph_0, 1)$. On the other hand, Theorem 4 easily implies that all the graphs (c)–(e) are simple.

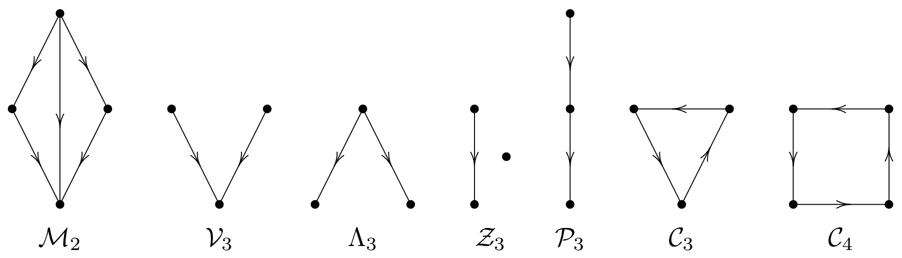


FIGURE 2. Seven small digraphs

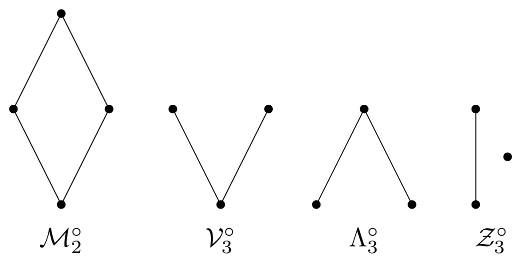


FIGURE 3. Hasse diagrams of four small posets

4. Simple homogeneous digraphs

In this section, in the course of establishing a simplicity criterion for homogeneous digraphs, we make use of seven distinguished digraphs given in Fig. 2.

For a digraph $\mathcal{D} = (V, E)$ let $\mathcal{D}^\circ = (V, E \cup \{(x, x) : x \in V\})$ denote the *reflexive closure* of $\mathcal{D}$. If $\mathcal{D}$ is a transitive digraph, then $\mathcal{D}^\circ$ is a poset. In particular, $\mathcal{M}_2^\circ$, $\mathcal{V}_3^\circ$, Λ_3° and $\mathcal{Z}_3^\circ$ are posets; their Hasse diagrams are depicted in Fig. 3. Also, notice that $\mathcal{D}$ is a homogeneous transitive digraph if and only if $\mathcal{D}^\circ$ is a homogeneous poset.

In the following we consider connected homogeneous digraphs with the property that each proper interval is non-null.

Lemma 5. *Let $\mathcal{D}$ be a homogeneous digraph. Then $\mathcal{D}$ has no proper null intervals if and only if either $\mathcal{D}$ is a tournament, or $\mathcal{Z}_3 \leqslant \mathcal{D}$, or $\mathcal{P}_3 \leqslant \mathcal{D}$.*

Proof. ($\Leftarrow$): A tournament obviously cannot have a null interval. Assume that $\mathcal{Z}_3 \leqslant \mathcal{D}$, and let I be a proper subset of $V(\mathcal{D})$ such that $E(\mathcal{D}[I]) = \varnothing$. Take any $x, y \in I$ such that $x \neq y$. Then $x \not\sim y$. Due to $\mathcal{Z}_3 \leqslant \mathcal{D}$ and homogeneity of $\mathcal{D}$ there is a vertex $z \in V(\mathcal{D})$ such that $z \to x$ and $z \not\sim y$. Clearly, $z \notin I$ because $z \to x$ and $E(\mathcal{D}[I]) = \varnothing$. But then I cannot be an interval in $\mathcal{D}$ because $z \to x$ and $z \not\sim y$. The proof is analogous in the case when $\mathcal{P}_3 \leqslant \mathcal{D}$.

($\Rightarrow$): Assume that $\mathcal{D}$ is not a tournament, and that $\mathcal{Z}_3 \not\leqslant \mathcal{D}$ and $\mathcal{P}_3 \not\leqslant \mathcal{D}$. Since $\mathcal{D}$ is not a tournament, there exist $x, y \in V(\mathcal{D})$ such that $x \neq y$ and $x \not\sim y$. Let us show that $I = \{x, y\}$ is an interval in $\mathcal{D}$. Take any $z \in V(\mathcal{D}) \setminus I$. Since $\mathcal{Z}_3 \not\leqslant \mathcal{D}$ and $\mathcal{P}_3 \not\leqslant \mathcal{D}$ we have that either $z \not\sim x$ and $z \not\sim y$, or $z \rightrightarrows I$, or $I \to z$. Therefore, I is an interval in $\mathcal{D}$. $\square$

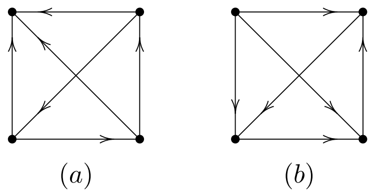


Figure 4. Two digraphs from Lemma 6

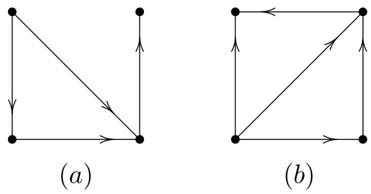


Figure 5. Two digraphs from Lemma 7

Lemma 6. *Let $\mathcal{D}$ be a connected homogeneous digraph with no proper null intervals. If $\mathcal{D}$ embeds each of the two digraphs (a) and (b) from Fig. 4 then $\mathcal{D}$ is simple.*

Proof. Let I be a proper interval in $\mathcal{D}$. Then $E(\mathcal{D}[I]) \neq \varnothing$ because $\mathcal{D}$ has no null intervals, so take any $x, y \in I$ such that $x \to y$. On the other hand, $E_{\mathcal{D}}(I, V \setminus I) \neq \varnothing$ because $\mathcal{D}$ is connected and I is proper. Therefore, for every $z \in V \setminus I$ we have that $z \rightrightarrows \{x, y\}$ or $\{x, y\} \rightrightarrows z$. Assume that $z \rightrightarrows \{x, y\}$. Then the homogeneity of $\mathcal{D}$ together with the fact that $\mathcal{D}$ embeds the configuration (a) yields that there is a vertex $u \in V(\mathcal{D})$ such that $x \to u \rightrightarrows \{y, z\}$. Now $u \notin I$ because $x \in I$ and $u \to z \to x$, while $u \notin V \setminus I$ because $x, y \in I$ and $x \to u \to y$. Contradiction.

The case $\{x, y\} \rightrightarrows z$ leads to the contradiction analogously, using the fact that $\mathcal{D}$ embeds configuration (b). □

Lemma 7. *Let $\mathcal{D}$ be a connected homogeneous digraph with no proper null intervals. If $\mathcal{D}$ embeds each of the two digraphs (a) and (b) from Fig. 5 then $\mathcal{D}$ is simple.*

Proof. Let I be a proper interval in D. Then $E(\mathcal{D}[I]) \neq \varnothing$ because $\mathcal{D}$ has no null intervals, so take any $x, y \in I$ such that $x \to y$. On the other hand, $E_{\mathcal{D}}(I, V \setminus I) \neq \varnothing$ because $\mathcal{D}$ is connected and I is proper. Therefore, for every $z \in V \setminus I$ we have that $z \rightrightarrows \{x, y\}$ or $\{x, y\} \rightrightarrows z$. Assume that $z \rightrightarrows \{x, y\}$. Then the homogeneity of $\mathcal{D}$ together with the fact that $\mathcal{D}$ embeds the configuration (a) yields that there is a vertex $u \in V(\mathcal{D})$ such that $y \to u$, $x \not\sim u$ and $z \not\sim u$. Now $u \notin I$ because $y \in I$ and $u \not\sim z \to y$, while $u \notin V \setminus I$ because $x, y \in I$ and $y \to u \not\sim x$. Contradiction.

The case $\{x, y\} \rightrightarrows z$ leads to the contradiction analogously, using the fact that $\mathcal{D}$ embeds configuration (b). □

Proposition 8. *Let $\mathcal{D}$ be a connected homogeneous digraph with no proper null intervals. If $\mathcal{D}$ has a cycle then $\mathcal{D}$ is simple.*

Proof. Assume first that $\mathcal{C}_3 \leqslant \mathcal{D}$ and let I be a proper interval in $\mathcal{D}$. Then $E(\mathcal{D}[I]) \neq \varnothing$ because $\mathcal{D}$ has no null intervals, so take any $x, y \in I$ such that

$x \to y$. On the other hand, $E_{\mathcal{D}}(I, V \setminus I) \neq \varnothing$ because $\mathcal{D}$ is connected and I is proper. Therefore, for every $z \in V \setminus I$ we have that $z \rightrightarrows \{x, y\}$ or $\{x, y\} \rightrightarrows z$. Assume that $z \rightrightarrows \{x, y\}$ (the other case is analogous). Because $\mathcal{D}$ is homogeneous and $\mathcal{C}_3 \leqslant \mathcal{D}$, there is a vertex $u \in V(\mathcal{D})$ such that $x \to u \to z \to x$. From $u \to z \to x$ it follows that $u \notin I$, whence $y \to u$. By the same argument, there is a vertex $v \in V(\mathcal{D})$ such that $x \to y \to v \to x$. Then $y \to v \to x$ forces $v \in I$, so $v \to u$ and $z \to v$. Now $\mathcal{D}[x, y, u, v]$ and $\mathcal{D}[x, y, z, v]$ are isomorphic to configurations (a) and (b) from Lemma 6, respectively. This contradicts the fact that there is a proper interval I in $\mathcal{D}$.

Assume now that $\mathcal{C}_3 \not\leqslant \mathcal{D}$. Let us first show that $\mathcal{C}_4 \leqslant \mathcal{D}$. Let n be the least integer such that there exist $x_1, \dots, x_n \in V(\mathcal{D})$ with $x_1 \to x_2 \to \dots \to x_n \to x_1$. Clearly, $n \geqslant 4$ and $x_1 \not\sim x_3$ due to the minimality of n. The mapping $f : \begin{pmatrix} x_1 & x_3 \\ x_3 & x_1 \end{pmatrix}$ is a local isomorphism, so it extends to an automorphism f^* of $\mathcal{D}$. Let $y = f^*(x_2)$. Then $x_1 \to x_2 \to x_3 \to y \to x_1$ and $x_2 \not\sim y$ because $\mathcal{C}_3 \not\leqslant \mathcal{D}$. Therefore, $\mathcal{C}_4 \leqslant \mathcal{D}$.

Let I be a proper interval in $\mathcal{D}$. Then $E(\mathcal{D}[I]) \neq \varnothing$ because $\mathcal{D}$ has no null intervals, so take any $x, y \in I$ such that $x \to y$. On the other hand, $E_{\mathcal{D}}(I, V \setminus I) \neq \varnothing$ because $\mathcal{D}$ is connected and I is proper. Therefore, for every $z \in V \setminus I$ we have that $z \rightrightarrows \{x, y\}$ or $\{x, y\} \rightrightarrows z$. Assume that $z \rightrightarrows \{x, y\}$ (the other case is analogous). Because $\mathcal{D}$ is homogeneous and $\mathcal{C}_4 \leqslant \mathcal{D}$, there exist $u, v \in V(\mathcal{D})$ such that $x \to y \to u \to v \to x$, $x \not\sim u$ and $y \not\sim v$. It is easy to see that $u, v \in I$, so $z \to u$ and $z \to v$. By the same argument, there exist $w, s \in V(\mathcal{D})$ such that $z \to y \to w \to s \to z$, $y \not\sim s$ and $w \not\sim z$. This time it is easy to see that $u, v \notin I$, so $\{x, u, v\} \rightrightarrows w$ and $s \not\sim x$, $s \not\sim y$, $s \not\sim u$ and $s \not\sim v$. Now $\mathcal{D}[x, y, z, u]$ and $\mathcal{D}[u, v, w, s]$ are isomorphic to configurations (a) and (b) from Lemma 7, respectively. This contradicts the fact that there is a proper interval I in $\mathcal{D}$. □

Lemma 9. *Let $\mathcal{D} = (V, E)$ be a connected homogeneous digraph with no proper null intervals. If $\mathcal{D}$ is not simple, then $\mathcal{D}^\circ$ is a homogeneous poset.*

Proof. Clearly, $\mathcal{D}^\circ$ is homogeneous, reflexive and antisymmetric. We know from Proposition 8 that $\mathcal{D}$ is acyclic. Assume that for some $x, y, z \in V(\mathcal{D})$ we have $x \to y$ and $y \to z$, but not $x \to z$. If $z \to x$ we have that $\mathcal{D}[x, y, z] \cong \mathcal{C}_3$, which contradicts the acyclicity of $\mathcal{D}$. If, however, $x \not\sim z$ then $\mathcal{C}_4 \leqslant \mathcal{D}$ as in the proof of Proposition 8, which contradicts the acyclicity of $\mathcal{D}$. Therefore, $\mathcal{D}^\circ$ must be transitive. □

A partially ordered set $(V, \leqslant)$ is a *tree* if $\downarrow x = \{y \in V : y \leqslant x\}$ is a chain for every $x \in V$, and it is a *dual tree* if $\uparrow x = \{y \in V : y \geqslant x\}$ is a chain for every $x \in V$.

Lemma 10. *No homogeneous tree (dual tree) is semisimple. In particular, no homogeneous chain is semisimple.*

Proof. Let $\mathcal{D} = (V, \leqslant)$ be a homogeneous tree, let $x \in V$ be arbitrary and let $I = \uparrow x$. Then I is a proper subset of V due to homogeneity of $\mathcal{D}$. Moreover, I is easily seen to be an interval in $\mathcal{D}$ (if $y < x$ then $y < i$ for all $i \in I$, while in case when $y \not\sim x$ we have that $y \not\sim i$ for every $i \in I$), and, moreover, a non-null one. Therefore, $\mathcal{D}$ is not semisimple. □

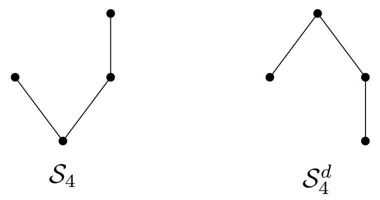


FIGURE 6. Two posets from the proof of Proposition 11

Proposition 11. *Let $\mathcal{D} = (V, \leqslant)$ be a connected homogeneous poset with no proper null intervals. Assume, further, that $\mathcal{D}$ is neither a tree nor a dual tree, and that it has a pair of incomparable elements. Then $\mathcal{D}$ is simple.*

Proof. Since $\mathcal{D}$ is connected with a pair of incomparable elements, it embeds $\mathcal{V}_3$ or Λ_3 or both. If $\mathcal{D}$ does not embed Λ_3 then it has to be a tree – a contradiction with the previous lemma. Analogously, if $\mathcal{D}$ does not embed $\mathcal{V}_3$ then it has to be a dual tree – a contradiction again. Therefore, $\mathcal{D}$ embeds both $\mathcal{V}_3$ and Λ_3.

Since $\mathcal{D}$ has no proper null intervals, Lemma 5 implies that $\mathcal{Z}_3 \leqslant \mathcal{D}$ ($\mathcal{D}$ is not a tournament because it has a pair of incomparable elements, while $\mathcal{P}_3 \nleqslant \mathcal{D}$ because $\mathcal{P}_3$ is not transitive). By amalgamating $\mathcal{Z}_3$ 'above' $\mathcal{V}_3$ and 'under' Λ_3 we obtain that $\mathcal{D}$ embeds both $\mathcal{S}_4$ and $\mathcal{S}_4^d$ given in Fig. 6

Assume now that I is a proper interval in $\mathcal{D}$. Then I is not a null interval, so there exist $x, y \in I$ such that $x < y$. Since $\mathcal{D}$ is connected, there exists a $z \in V \setminus I$ such that $z > x \wedge z > y$, or $z < x \wedge z < y$. Assume that $z > x \wedge z > y$ (the other case follows by dual arguments). As we have just seen, $\mathcal{D}$ embeds $\mathcal{S}_4$, so by the homogeneity of $\mathcal{D}$ there exists a vertex $u \in V$ such that $u > x$, $u \not\sim y$ and $u \not\sim z$. Now $u \notin I$ because $z > y$ and $z \not\sim u$, while $u \notin V \setminus I$ because $u > x$ and $u \not\sim y$. A contradiction. □

Bearing in mind Schmerl's characterisation of countable homogeneous posets [13], we have the following situation regarding their simplicity:

- an antichain is semisimple but not simple (unless it is trivial);
- an antichain of proper chains is not semisimple;
- a chain of antichains is not semisimple;
- the generic (random) poset $\mathcal{P}$ (the unique countable homogeneous poset whose age contains all finite posets) is simple.

We are now ready to state and prove the principal result of this section.

Theorem 12. *Let $\mathcal{D}$ be a homogeneous digraph. Then $\mathcal{D}$ is simple if and only if*

- *$\mathcal{D}$ is a tournament and $\mathcal{C}_3 \leqslant \mathcal{D}$; or*
- *the following holds:*
  - *$\mathcal{D}$ is connected,*
  - *$\mathcal{Z}_3 \leqslant \mathcal{D}$ or $\mathcal{P}_3 \leqslant \mathcal{D}$, and*
  - *if $\mathcal{D}$ is acyclic then $\mathcal{M}_2 \leqslant \mathcal{D}$.*

In particular, a homogeneous tournament $\mathcal{D}$ is simple if and only if $\mathcal{C}_3 \leqslant \mathcal{D}$.

Proof. ($\Rightarrow$): Let $\mathcal{D}$ be a simple homogeneous digraph. If $\mathcal{D}$ is a tournament but $\mathcal{C}_3 \nleqslant \mathcal{D}$ then $\mathcal{D}$ is acyclic, hence transitive. Therefore, $\mathcal{D}^\circ$ is a homogeneous chain, and homogeneous chains are not simple.

Assume now that $\mathcal{D}$ is not a tournament. If $\mathcal{D}$ is not connected, then $\mathcal{D}$ cannot be simple. If $\mathcal{D}$ is connected, but $\mathcal{Z}_3 \nleqslant \mathcal{D}$ and $\mathcal{P}_3 \nleqslant \mathcal{D}$, then $\mathcal{D}$ has null intervals by Lemma 5, so it cannot be simple. Finally, assume that $\mathcal{D}$ is connected, that $\mathcal{Z}_3 \leqslant \mathcal{D}$ or $\mathcal{P}_3 \leqslant \mathcal{D}$ (hence, $\mathcal{D}$ has no null intervals), that $\mathcal{D}$ is acyclic, but $\mathcal{M}_2 \nleqslant \mathcal{D}$. Then, due to homogeneity, $\mathcal{D}^\circ$ is a poset, and $\mathcal{V}_3 \nleqslant \mathcal{D}$ or $\Lambda_3 \nleqslant \mathcal{D}$. Therefore, $\mathcal{D}$ is a tree or a dual tree, and these are not simple.

($\Leftarrow$): Let $\mathcal{D}$ be a homogeneous digraph. If $\mathcal{D}$ is a tournament and $\mathcal{C}_3 \leqslant \mathcal{D}$ then it is simple by Proposition 8.

Assume now that $\mathcal{D}$ is not a tournament, but that it is connected and has no null intervals (which, according to Lemma 5, follows from $\mathcal{Z}_3 \leqslant \mathcal{D}$ or $\mathcal{P}_3 \leqslant \mathcal{D}$). If $\mathcal{D}$ has a cycle then $\mathcal{D}$ is simple by Proposition 8. If, however, $\mathcal{D}$ is acyclic then, by the assumption, $\mathcal{M}_2 \leqslant \mathcal{D}$. If $\mathcal{D}^\circ$ is not a poset, then $\mathcal{D}$ is simple by Lemma 9. Finally, if $\mathcal{D}^\circ$ is a poset, then $\mathcal{D}^\circ$ is neither a tree nor a dual tree because of $\mathcal{M}_2 \leqslant \mathcal{D}$, so it is simple by Proposition 11. □

Going back to the countable case, by Cherlin's catalogue of countable homogeneous digraphs given at the beginning of [4, Chapter 5], we have that $\mathbb{Q}$ is the only non-simple countable homogeneous tournament of the five such tournaments found initially by Lachlan in [11]. The other four are simple, namely: the trivial tournament, the three-element cycle $\mathcal{C}_3$, the random tournament $\mathcal{T}^\infty$, and the dense local order $\mathcal{S}(2)$.

5. Semisimple homogeneous graphs and digraphs

In this section we consider connected homogeneous graphs and digraphs which contain null intervals, striving to single out the semisimple ones. Clearly, if I is a null interval in D and $J \subseteq I$, $|J| \geqslant 2$, then J is also a null interval. Therefore, maximal (with respect to set inclusion) null intervals are of interest. Related results are contained in Földes [7].

Lemma 13. *Let $\mathcal{G}$ be a connected homogeneous (di)graph.*

- (a) *If I and J are distinct maximal null intervals in $\mathcal{G}$ then $I \cap J = \varnothing$.*
- (b) *If I and J are distinct maximal null intervals in $\mathcal{G}$ then $|I| = |J|$.*
- (c) *The set of all maximal null intervals in $\mathcal{G}$ is a partition of $V(\mathcal{G})$.*

Proof. (a) Assume that $x \in I \cap J$.

Let us show that $I \cup J$ is a proper interval in $\mathcal{G}$. Clearly $|I \cup J| \geqslant 2$. Assume that $I \cup J = V(\mathcal{G})$. Take any $y \in I \setminus J$ and any $z \in J \setminus I$. Then $x \not\sim y$ since $E(\mathcal{G}[I]) = \varnothing$, whence follows that $y \not\sim z$ because J is an interval and $y \notin J$. This implies $E(\mathcal{G}) = \varnothing$, which contradicts the fact that $\mathcal{G}$ is connected. So, $I \cup J \subset V(\mathcal{G})$.

The same argument ensures that $E(\mathcal{G}[I \cup J]) = \varnothing$. Finally, let us show that $I \cup J$ is an interval. Let $u \notin I \cup J$ be arbitrary. If $u \rightrightarrows I$ then $u \to x$, so $u \rightrightarrows J$, because J is an interval (in the undirected case $u \sim I$ implies $u \sim x$ and so $u \sim J$). Other cases follow analogously.

Therefore, $I \cup J$ is a null (and hence proper) interval in $\mathcal{G}$. This is a contradiction with the maximality of I and J.

(b) Take any $x \in I$, any $y \in J$ and let $f : \{x\} \to \{y\}$. Then f is a partial automorphism, so, due to homogeneity of $\mathcal{G}$, f extends to $f^* \in \mathrm{Aut}(\mathcal{G})$. Clearly, $f^*(I)$ is a null interval in $\mathcal{G}$ which intersects J, so $f^*(I) \subseteq J$ due to maximality of J. Therefore, $|I| \leqslant |J|$. Analogously, $|J| \leqslant |I|$.

(c) Since there exist null intervals in $\mathcal{G}$, there exist (by Zorn's Lemma) maximal null intervals in $\mathcal{G}$; let I be one of them. Take any $x \in I$ and $y \notin I$. Then $f : \{x\} \to \{y\}$ is a partial automorphism which extends to $f^* \in \mathrm{Aut}(\mathcal{G})$, and $f^*(I)$ is a null interval that contains y. Therefore, every vertex of $\mathcal{G}$ belongs to a null interval in $\mathcal{G}$, and hence, to a maximal null interval in $\mathcal{G}$. We already know that distinct maximal null intervals in $\mathcal{G}$ have to be disjoint. □

Let $\theta_\mathcal{G}$ be an equivalence relation on $V(\mathcal{G})$ whose blocks are the maximal null intervals in $\mathcal{G}$. Define the factor $\mathcal{G}/\theta_\mathcal{G}$ in a natural way: the set of vertices is $V(\mathcal{G})/\theta_\mathcal{G}$, and $I \sim J$ (resp. $I \to J$) in $\mathcal{G}/\theta_\mathcal{G}$ if and only if $x \sim y$ (resp. $x \to y$) in $\mathcal{G}$ for some $x \in I$ and $y \in J$. The definition is correct because I and J are disjoint intervals.

Lemma 14. *Let $\mathcal{G}$ be a connected homogeneous (di)graph containing null intervals.*

- (a) *Let J be a union of some $\theta_\mathcal{G}$ classes. Then J is an interval in $\mathcal{G}$ if and only if $J/\theta_\mathcal{G}$ is an interval in $\mathcal{G}/\theta_\mathcal{G}$.*
- (b) *Every proper interval in $\mathcal{G}/\theta_\mathcal{G}$ is non-null.*
- (c) *$\mathcal{G}$ is semisimple if and only if $\mathcal{G}/\theta_\mathcal{G}$ is simple.*

Proof. (a) is easy. (b) follows from (a) and the fact that the classes of $\theta_\mathcal{G}$ are maximal null intervals in $\mathcal{G}$. Finally, (c) follows from (a) and (b). □

The statement (c) from the previous lemma can be significantly strengthened.

Lemma 15. *Let $\mathcal{G}$ be a connected homogeneous (di)graph containing proper null intervals. Then $\mathcal{G}/\theta_\mathcal{G}$ is a complete graph in the undirected case, and a tournament in the directed case.*

Proof. Note first that Lemmas 2 and 5 imply that $\mathcal{Z}_3' \not\leqslant \mathcal{G}$ in the undirected case, while we have $\mathcal{Z}_3 \not\leqslant \mathcal{G}$ and $\mathcal{P}_3 \not\leqslant \mathcal{G}$ if $\mathcal{G}$ is a digraph.

Assume that $\mathcal{G}/\theta_\mathcal{G}$ is not a complete graph (resp. not a tournament). Then there exist $I, J \in V(\mathcal{G}/\theta_\mathcal{G})$ such that $I \neq J$ and $I \not\sim J$ in $\mathcal{G}/\theta_\mathcal{G}$. Then I and J are distinct maximal null intervals in $\mathcal{G}$ and $x \not\sim y$ in $\mathcal{G}$ for all $x \in I$ and $y \in J$. In order to reach a contradiction, it suffices to show that $I \cup J$ is a null interval in $\mathcal{G}$, since this contradicts the maximality of I.

Note that $I \cup J$ is a proper subset of $\mathcal{G}$ since $\mathcal{G}$ is connected and $E(\mathcal{G}[I \cup J]) = \varnothing$. Take any $x \in V(\mathcal{G}) \setminus (I \cup J)$ and any $y \in I \cup J$. If $x \not\sim y$ then $x \not\sim z$ for all $z \in I \cup J$ because $\mathcal{Z}_3' \not\leqslant \mathcal{G}$ (resp. $\mathcal{Z}_3 \not\leqslant \mathcal{G}$). In the undirected case, if $x \sim y$ then we must have $x \sim z$ for all $z \in I$ in order to avoid $\mathcal{Z}_3'$ as an induced subgraph of $\mathcal{G}$. Similarly, if $\mathcal{G}$ is a digraph and $x \to y$, then $x \to z$ for all $z \in I \cup J$ because $Z_3 \not\leqslant \mathcal{G}$ and $P_3 \not\leqslant \mathcal{G}$. By the same argument, if $y \to x$ then $z \to x$ for all $z \in I \cup J$. Therefore, $I \cup J$ is a null interval in $\mathcal{G}$. □

For undirected graphs, this suffices to characterise the 'joint effect' of semisimplicity and homogeneity.

Theorem 16. *Let $\mathcal{G}$ be a homogeneous graph. Then $\mathcal{G}$ is semisimple if and only if one of the following holds:*

- *$\mathcal{G}$ is isomorphic to $\mathcal{K}_1$; or*
- *$\mathcal{G}$ is isomorphic to $\overline{\mu\mathcal{K}_\nu}$ for some cardinals $\mu > 0$, $\nu > 1$; or*
- *$\mathcal{G}$ is connected and $\mathcal{Z}_3' \leqslant \mathcal{G}$.*

Proof. (⇒): If $\mathcal{G}$ is disconnected then $\mathcal{G}$ must contain no edges, so it is isomorphic to $\overline{\mathcal{K}_\nu}$ for some ν. Otherwise, if $\mathcal{G}$ is connected and semisimple, then it is either simple, so that $\mathcal{Z}_3' \leqslant \mathcal{G}$ follows by Theorem 4, or contains proper null intervals. In the latter case, $\mathcal{G}/\theta_\mathcal{G}$ is a complete graph, while the blocks of $\theta_\mathcal{G}$ must be null subgraphs with a fixed number of $\nu > 1$ vertices. If $|\mathcal{G}/\theta_\mathcal{G}| = \mu$, we have $\mathcal{G} \cong \overline{\mu\mathcal{K}_\nu}$.

(⇐): This is immediate, bearing in mind Theorem 4. □

In the case of digraphs, the following fully clarifies the situation.

Proposition 17. *Let $\mathcal{D}$ be a homogeneous digraph containing proper null intervals. Then $\mathcal{D}$ is semisimple if and only if either $E(\mathcal{D}) = \varnothing$, or $\mathcal{D}$ is connected and $\mathcal{C}_3 \leqslant \mathcal{D}$.*

Proof. (⇒): Assume that $\mathcal{D}$ is semisimple. If $\mathcal{D}$ is not connected then $E(\mathcal{D}) = \varnothing$ by Lemma 1. If $\mathcal{D}$ is connected then $\mathcal{D}/\theta_\mathcal{D}$ is a simple tournament (Lemmas 14 and 15), so $\mathcal{C}_3 \leqslant \mathcal{D}$ by Theorem 12.

(⇐): If $E(\mathcal{D}) = \varnothing$ then $\mathcal{D}$ is clearly semisimple. Assume now that $\mathcal{D}$ is connected and $\mathcal{C}_3 \leqslant \mathcal{D}$. Then $\mathcal{C}_3 \leqslant \mathcal{D}/\theta_\mathcal{D}$. Therefore, $\mathcal{D}/\theta_\mathcal{D}$ is a simple tournament (Lemma 15 and Theorem 12), whence it follows that $\mathcal{D}$ is semisimple (Lemma 14). □

The final result of this note, characterising semisimple homogeneous digraphs, now follows immediately.

Theorem 18. *Let $\mathcal{D}$ be a homogeneous digraph. Then $\mathcal{D}$ is semisimple if and only if*

- *$E(\mathcal{D}) = \varnothing$; or*
- *$\mathcal{D}$ is connected, $\mathcal{Z}_3 \nleqslant \mathcal{D}$, $\mathcal{P}_3 \nleqslant \mathcal{D}$ and $C_3 \leqslant \mathcal{D}$; or*
- *$\mathcal{D}$ satisfies the following:*
  - *$\mathcal{D}$ is connected,*
  - *$\mathcal{Z}_3 \leqslant \mathcal{D}$ or $\mathcal{P}_3 \leqslant \mathcal{D}$, and*
  - *if $\mathcal{D}$ is acyclic then $\mathcal{M}_2 \leqslant \mathcal{D}$.*

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