

# On free spectra of finite monoids from the pseudovariety DA

Igor Dolinka

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**Abstract** The pseudovariety DA consists of all aperiodic finite monoids all of whose regular  $\mathcal{D}$ -classes are subsemigroups (that is, rectangular subbands); this pseudovariety appears quite frequently in various contexts in finite semigroup theory. In this note we prove that all its members have a log-polynomial free spectrum, thereby making a new step towards proving the Seif conjecture on the dichotomy of free spectra of finite monoids.

**Keywords** Free spectra · Bilateral semidirect product · Pseudovariety · Finite monoid · Semilattice

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## 1 Introduction

Let  $\mathcal{V}$  be a variety of algebraic structures and let  $F_n(\mathcal{V})$  denote the  $\mathcal{V}$ -free algebra over an  $n$ -element set,  $n \geq 1$ . If  $\mathcal{V}$  is locally finite, then each finitely generated algebra in  $\mathcal{V}$  is finite, so the sequence  $f_n(\mathcal{V}) = |F_n(\mathcal{V})|$  consists of positive integers. This sequence is called the *free spectrum* of  $\mathcal{V}$ . The free spectrum of an individual algebra  $A$  is just the free spectrum of the variety generated by  $A$ . The ultimate goal of investigations of free spectra is determining all possible sequences (at least asymptotically) and thus obtaining a corresponding classification of finite algebras and locally finite varieties. An overview of basic results and development of the theory up to the early nineties may be found in [8].

The establishment of tame congruence theory [10] as a key component of finite universal algebra also yielded new results on the behavior free spectra of finite algebras [3]. However, the crucial impetus for further research came from the paper of Kearnes [14] who showed that the free spectrum of a general finite algebra is a great deal governed by the free spectrum of an associated monoid, called the *twin monoid*. This clearly puts forward the issue of

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I. Dolinka  
Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4,  
21101 Novi Sad, Serbia  
E-mail: dockie@dmi.uns.ac.rs

studying free spectra of semigroups and monoids. As a first approximation to the classification programme mentioned above, Seif [17] proposed an intriguing conjecture concerning a dichotomy in the asymptotical behavior of free spectra of finite monoids. Namely, we say that an algebra  $A$  has a *doubly exponential* free spectrum if there is a real number  $c > 0$  such that

$$f_n(A) \geq 2^{2^{cn}}$$

holds for large  $n$ . (This is precisely as far as the free spectrum of a finite algebra can go, since by elementary universal-algebraic arguments we have  $f_n(A) \leq |A|^{|A|^n}$ .) Otherwise, the free spectrum of  $A$  is *sub-log-exponential*; moreover, it is *log-polynomial* (or *small*) if there is an integer  $k \geq 1$  such that  $\log f_n(A) \in \mathcal{O}(n^k)$  (here and in the whole paper ‘log’ refers to the base-2 logarithm, although this is not essential as all logarithms are asymptotically equivalent). For example, an old result of Higman [9] and Neumann [15] states that a finite group  $G$  is nilpotent of step  $c$  if and only if  $\log f_n(G) \in \mathcal{O}(n^c)$ ; otherwise, if  $G$  is not nilpotent, then  $f_n(G)$  is doubly exponential. Seif’s conjecture attempts to demarcate between finite monoids whose free spectra are sub-log-exponential and doubly exponential, respectively.

To formulate this conjecture explicitly, we need to introduce several pseudovarieties of finite monoids, that is, classes of finite monoids closed under taking homomorphic images, submonoids and finite direct products. (We refer to [2] for basic notions on semigroups and pseudovarieties.) First of all, let  $\text{DS}$  be the pseudovariety consisting of all finite monoids  $M$  with the property that all of its regular  $\mathcal{D}$ -classes are subsemigroups of  $M$ . A monoid is called *aperiodic* (or *combinatorial*) if all of its subgroups are trivial. All aperiodic members of  $\text{DS}$  again form a pseudovariety, which is denoted by  $\text{DA}$ ; it is characterised by the condition that each regular  $\mathcal{D}$ -class is a rectangular band. Now, for a monoid pseudovariety  $\mathbf{V}$  we denote by  $\text{EV}$  the pseudovariety consisting of all monoids  $M$  such that  $\langle E(M) \rangle$ , the submonoid of  $M$  generated by the set  $E(M)$  of all idempotents of  $M$ , belongs to  $\mathbf{V}$ . Furthermore, for a pseudovariety  $\mathbf{H}$  of finite groups, we denote by  $\overline{\mathbf{H}}$  the pseudovariety of all finite monoids all of whose subgroups belong to  $\mathbf{H}$ . Finally, let  $\mathbf{G}_{\text{nil}}$  denote the pseudovariety of all finite nilpotent groups. The conjecture of Seif asserts that for a monoid  $M$ ,  $f_n(M)$  is sub-log-exponential if and only if  $M \in \text{EDA} \cap \mathbf{G}_{\text{nil}}$ . In [17], Seif himself proved the direct part of this conjecture and verified its converse for all finite monoids obtained by adjoining an identity element to a Rees matrix semigroup. The present author showed in [6] that the conjecture holds for monoids that are unions of its subgroups: a finite completely regular semigroup has a sub-log-exponential (moreover, a small) free spectrum if and only if it is locally orthodox and has only nilpotent subgroups. See [5, 7, 12, 13, 16] for further background.

In this note we prove the following result.

**Theorem 1**  $f_n(M)$  is log-polynomial for each finite monoid  $M \in \text{DA}$ .

To achieve this, we employ the result of Straubing and Thérien [19] (see also [20]), according to which  $\text{DA} = \bigcup_{t \geq 1} \text{Sl}^{(t)}$ , where  $\text{Sl}^{(t)}$  is the pseudovariety generated by all left-associated,  $t$  times iterated bilateral semidirect products of semilattice monoids. Then we use the fact, proved by Blanchet-Sadri [4], that each pseudovariety  $\text{Sl}^{(t)}$  is equational, i.e. that there is a semigroup variety  $\mathcal{U}_t$  such that  $\text{Sl}^{(t)}$  is precisely the class of finite members of  $\mathcal{U}_t$ . In particular, this means that  $\text{DA}$  is contained in  $\bigcup_{t \geq 1} \mathcal{U}_t$ ; so, our result follows immediately as soon as we prove, for each  $t \geq 1$ , that  $\log f_n(\mathcal{U}_t)$  is bounded above by a polynomial function of  $n$ . This approach roughly parallels (and in several points generalises) that of Horváth et al. [11] who utilised the results of Almeida [1, 2] and Stiffler [18] to obtain an analogous result for finite  $\mathcal{R}$ -trivial (and, dually,  $\mathcal{L}$ -trivial) monoids, which form subpseudovarieties  $\mathbf{R}$  and  $\mathbf{L}$ , respectively, of  $\text{DA}$ .

In the following section, we gather all the necessary notions and prerequisites for our proof. Then, in Sect. 3, we introduce a sequence of congruences on the free monoid on a countably infinite alphabet, and this sequence will contain the fully invariant congruences corresponding to  $\mathcal{U}_t$ . This leads to an inequality used in estimating free spectra of varieties  $\mathcal{U}_t$ ,  $t \geq 1$ . The final calculation of asymptotical upper bounds of these free spectra is performed in Sect. 4.

## 2 Preliminaries

Let  $S$  and  $T$  be semigroups; as it is usual in the present context, the operation of  $S$  will be written additively (which does not mean, however, that  $S$  is commutative), while that of  $T$  will be written multiplicatively. A *left action* of  $T$  on  $S$  is a mapping  $T \times S \rightarrow S$ , written  $(t, s) \mapsto ts$ , such that

$$t(s + s') = ts + ts' \quad \text{and} \quad (tt')s = t(t's)$$

holds for all  $s, s' \in S$  and  $t, t' \in T$ . If  $S$  and  $T$  are monoids, whose identity elements are 0 and 1, respectively, then we also require that  $t0 = 0$  and  $1s = s$  holds for all  $s \in S$ ,  $t \in T$ . The notion of a *right action* of  $T$  on  $S$  is defined dually.

Given a left action of  $T$  on  $S$ , one defines the *semidirect product*  $S * T$  (with respect to the considered action) on the set  $S \times T$  by

$$(s, t)(s', t') = (s + ts', tt')$$

for all  $s, s' \in S$ ,  $t, t' \in T$ . It is easy to check that  $S * T$  is again a semigroup; moreover, it is a monoid if both  $S$  and  $T$  are monoids ( $(0, 1)$  turns out to be the identity element of  $S * T$ ). If  $\mathcal{V}_1, \mathcal{V}_2$  are two pseudovarieties of finite monoids, then one can define their semidirect product  $\mathcal{V}_1 * \mathcal{V}_2$  as the pseudovariety consisting of all finite monoids that divide (= are homomorphic images of submonoids of) monoids of the form  $S * T$ , where  $S \in \mathcal{V}_1$ ,  $T \in \mathcal{V}_2$ . Even though the semidirect product is not associative on the level of individual semigroups, it is associative for pseudovarieties.

On the other hand, given two monoids  $S$  and  $T$ , one can fix two actions of  $T$  on  $S$ , one left and one right, and consider a ‘left-right symmetric version’ of the semidirect product that is called the *bilateral* (or *two-sided*) *semidirect product*, denoted by  $S ** T$ . This is a monoid whose underlying set is again  $S \times T$ , while its operation is defined by

$$(s, t)(s', t') = (st' + ts', tt')$$

for all  $s, s' \in S$ ,  $t, t' \in T$ ; as before,  $(0, 1)$  is easily verified to be the identity element. For two pseudovarieties of finite monoids  $\mathcal{V}_1, \mathcal{V}_2$ , we define  $\mathcal{V}_1 ** \mathcal{V}_2$  to be the pseudovariety consisting of all finite monoids dividing  $S ** T$  such that  $S \in \mathcal{V}_1$ ,  $T \in \mathcal{V}_2$ . Unlike the case of the ordinary semidirect product construction, the bilateral semidirect product of two pseudovarieties is not associative, so that the placement of parentheses becomes relevant: we have  $(\mathcal{V}_1 ** \mathcal{V}_2) ** \mathcal{V}_3 \subseteq \mathcal{V}_1 ** (\mathcal{V}_2 ** \mathcal{V}_3)$ , while the converse inclusion is in general not true.

So, let  $\text{Sl}$  be the pseudovariety of all finite semilattice monoids and define the sequence of pseudovarieties  $\text{Sl}^{(t)}$ ,  $t \geq 1$ , by  $\text{Sl}^{(1)} = \text{Sl}$  and  $\text{Sl}^{(t+1)} = \text{Sl}^{(t)} ** \text{Sl}$  for all  $t \geq 1$ . Therefore,  $\text{Sl}^{(t)}$  consists of all finite monoids dividing a bilateral semidirect product of the form

$$(\dots((Y_1 ** Y_2) ** Y_3)\dots) ** Y_t,$$

where  $Y_1, \dots, Y_t$  are semilattice monoids.

**Fact 2 (Straubing and Thérien [19])**

$$\bigcup_{t \geq 1} \text{Sl}^{(t)} = \text{DA}.$$

This is a ‘bilateral analogue’ of a well-known result of Stiffler [18], asserting that if we define a sequence  $\text{Sl}_t$  of pseudovarieties by  $\text{Sl}_1 = \text{Sl}$  and  $\text{Sl}_{t+1} = \text{Sl}_t * \text{Sl}$ , then we have  $\bigcup_{t \geq 1} \text{Sl}_t = \text{R}$ .

The paper of Blanchet-Sadri [4] provides a basis of pseudoidentities for each  $\text{Sl}^{(t)}$ . As it turns out, all these pseudovarieties are *equational*, i.e. they have bases consisting entirely of ordinary identities, which means that each of them is a finite trace (the class of finite members) of a monoid variety. In the following, for a word  $\mathbf{w} \in X^*$  let  $c(\mathbf{w})$  denote the *content* of  $\mathbf{w}$ , the set of all letters from  $X$  occurring in  $\mathbf{w}$ .

**Fact 3 (Blanchet-Sadri [4])** *Let the set  $D_t$ ,  $t \geq 1$ , consist of the following two types of identities over a countably infinite alphabet  $X$  containing the letters  $x, y$ :*

$$\mathbf{u}_{t-1} \dots \mathbf{u}_1 x^2 \mathbf{v}_1 \dots \mathbf{v}_{t-1} \doteq \mathbf{u}_{t-1} \dots \mathbf{u}_1 x \mathbf{v}_1 \dots \mathbf{v}_{t-1}, \quad (2.1)$$

where  $\mathbf{u}_i, \mathbf{v}_j \in X^+$ ,  $1 \leq i, j < t$ , are such that  $x \in c(\mathbf{u}_1) \cap c(\mathbf{v}_1)$  and  $c(\mathbf{u}_i \mathbf{v}_i) \subseteq c(\mathbf{u}_{i+1}) \cap c(\mathbf{v}_{i+1})$  for all  $1 \leq i < t-1$ , and

$$\mathbf{u}_{t-1} \dots \mathbf{u}_1 x y \mathbf{v}_1 \dots \mathbf{v}_{t-1} \doteq \mathbf{u}_{t-1} \dots \mathbf{u}_1 y x \mathbf{v}_1 \dots \mathbf{v}_{t-1}, \quad (2.2)$$

where  $\mathbf{u}_i, \mathbf{v}_j \in X^+$ ,  $1 \leq i, j < t$ , are such that  $x, y \in c(\mathbf{u}_1) \cap c(\mathbf{v}_1)$  and  $c(\mathbf{u}_i \mathbf{v}_i) \subseteq c(\mathbf{u}_{i+1}) \cap c(\mathbf{v}_{i+1})$  for all  $1 \leq i < t-1$ .

Then the pseudovariety  $\text{Sl}^{(t)}$  is defined by  $D_t$ .

In other words, if  $\mathcal{U}_t$  denotes the variety of monoids defined by all identities of the form (2.1) and (2.2), then  $\text{Sl}^{(t)}$  is just the class of all finite monoids belonging to  $\mathcal{U}_t$ . For this reason, any finite monoid from  $\text{DA}$  belongs to a variety  $\mathcal{U}_t$  for some  $t$  (by Fact 2); thus our Theorem 1 will be an immediate consequence of the following assertion.

**Theorem 4** *For each  $t \geq 1$  we have  $\log f_n(\mathcal{U}_t) \in \mathcal{O}(n^{2t-1})$ .*

We finish this preliminary section by invoking the ‘one-sided version’ of Fact 3, due to Almeida, which gives an equational axiomatisation for pseudovarieties generated by iterated semidirect products of semilattice monoids.

**Fact 5 (Almeida [1,2])** *Let the set  $\Sigma_t$ ,  $t \geq 1$ , consist of the following two types of identities over a countably infinite alphabet  $X$  containing the letters  $x, y$ :*

$$\mathbf{u}_{t-1} \dots \mathbf{u}_1 x^2 \doteq \mathbf{u}_{t-1} \dots \mathbf{u}_1 x, \quad (2.3)$$

where  $\mathbf{u}_i \in X^+$ ,  $1 \leq i < t$ , are such that  $x \in c(\mathbf{u}_1)$  and  $c(\mathbf{u}_i) \subseteq c(\mathbf{u}_{i+1})$  for all  $1 \leq i < t-1$ , and

$$\mathbf{u}_{t-1} \dots \mathbf{u}_1 x y \doteq \mathbf{u}_{t-1} \dots \mathbf{u}_1 y x, \quad (2.4)$$

where  $\mathbf{u}_i \in X^+$ ,  $1 \leq i < t$ , are such that  $x, y \in c(\mathbf{u}_1)$  and  $c(\mathbf{u}_i) \subseteq c(\mathbf{u}_{i+1})$  for all  $1 \leq i < t-1$ . Then the pseudovariety  $\text{Sl}_t$  is defined by  $\Sigma_t$ .

So,  $\text{Sl}_t$  is just the class of all finite members of monoid varieties  $\mathcal{V}_t$  defined by all identities of the form (2.3) and (2.4); in turn,  $\text{Sl}_t$  generates  $\mathcal{V}_t$  as a variety. Horváth et al. [11] proved that  $\log f_n(\mathcal{V}_t) \in \mathcal{O}(n^t)$ , which, in combination with Stiffler’s result, implies that any finite  $\mathcal{R}$ -trivial monoid has a log-polynomial free spectrum.

### 3 Some congruences on free monoids

As before, we let  $X$  to be a fixed countably infinite alphabet; all letters that are going to appear in our further considerations will belong to  $X$ . Recall that for a word  $\mathbf{w} \in X^*$ ,  $c(\mathbf{w}) \subset X$  is the set of all letters occurring in  $\mathbf{w}$ . By  $\sigma(\mathbf{w})$  we denote the letter from  $c(\mathbf{w})$  that is the last to appear in  $\mathbf{w}$  from the left; consequently, we have a unique factorisation  $\mathbf{w} = s(\mathbf{w})\sigma(\mathbf{w})\bar{s}(\mathbf{w})$ , where  $s(\mathbf{w})$  is the longest prefix of  $\mathbf{w}$  not containing  $\sigma(\mathbf{w})$  (i.e. with a content strictly smaller than  $\mathbf{w}$ ).

Inspired by identities appearing in results of Blanchet-Sadri and Almeida (Facts 3 and 5, respectively), we define a sequence of congruences  $\rho_{a,b}$  on the free monoid  $X^*$  for integers  $a, b \geq 1$ . Namely, if  $a \geq b$ , we let  $\rho_{a,b}$  to be congruence of  $X^*$  generated by all pairs of the form

$$(\mathbf{u}_{a-1} \dots \mathbf{u}_1 x^2 \mathbf{v}_1 \dots \mathbf{v}_{b-1}, \mathbf{u}_{a-1} \dots \mathbf{u}_1 x \mathbf{v}_1 \dots \mathbf{v}_{b-1}) \quad (3.1)$$

such that

- $x \in c(\mathbf{u}_1) \cap c(\mathbf{v}_1)$  if  $b > 1$  (this is replaced by  $x \in c(\mathbf{u}_1)$  if  $b = 1$ ),
- $c(\mathbf{u}_i \mathbf{v}_i) \subseteq c(\mathbf{u}_{i+1}) \cap c(\mathbf{v}_{i+1})$  for all  $1 \leq i < b-1$ ,
- $c(\mathbf{u}_{b-1} \mathbf{v}_{b-1}) \subseteq c(\mathbf{u}_b)$ , provided  $a > b$ , and
- $c(\mathbf{u}_i) \subseteq c(\mathbf{u}_{i+1})$  for all  $b \leq i < a-1$ ,

and all pairs of the form

$$(\mathbf{u}_{a-1} \dots \mathbf{u}_1 x y \mathbf{v}_1 \dots \mathbf{v}_{b-1}, \mathbf{u}_{a-1} \dots \mathbf{u}_1 y x \mathbf{v}_1 \dots \mathbf{v}_{b-1}) \quad (3.2)$$

such that

- $x, y \in c(\mathbf{u}_1) \cap c(\mathbf{v}_1)$  if  $b > 1$  (or  $x, y \in c(\mathbf{u}_1)$  if  $b = 1$ ),
- $c(\mathbf{u}_i \mathbf{v}_i) \subseteq c(\mathbf{u}_{i+1}) \cap c(\mathbf{v}_{i+1})$  for all  $1 \leq i < b-1$ ,
- $c(\mathbf{u}_{b-1} \mathbf{v}_{b-1}) \subseteq c(\mathbf{u}_b)$ , provided  $a > b$ , and
- $c(\mathbf{u}_i) \subseteq c(\mathbf{u}_{i+1})$  for all  $b \leq i < a-1$ .

If  $a < b$ , the congruence  $\rho_{a,b}$  is dual to  $\rho_{b,a}$  in the sense that  $(\mathbf{u}, \mathbf{v}) \in \rho_{a,b}$  if and only if  $(\overleftarrow{\mathbf{u}}, \overleftarrow{\mathbf{v}}) \in \rho_{b,a}$ , where  $\overleftarrow{\mathbf{w}}$  is the reverse word of  $\mathbf{w}$ .

We record two important properties of the congruences just introduced.

**Lemma 6** *All congruences  $\rho_{a,b}$  ( $a, b \geq 1$ ) are fully invariant.*

*Proof* Because of duality, there is no loss of generality in assuming that  $a \geq b$ . If  $(\mathbf{u}, \mathbf{v}) \in \rho_{a,b}$  then there is a sequence of words  $\mathbf{u} = \mathbf{w}_0, \dots, \mathbf{w}_m = \mathbf{v} \in X^*$  such that for each  $1 \leq k \leq m$  there are words  $\mathbf{p}_k, \mathbf{q}_k, \mathbf{w}'_{k-1}, \mathbf{w}''_k \in X^*$  satisfying

$$\begin{aligned} \mathbf{w}_{k-1} &= \mathbf{p}_k \mathbf{w}'_{k-1} \mathbf{q}_k, \\ \mathbf{w}_k &= \mathbf{p}_k \mathbf{w}''_k \mathbf{q}_k, \end{aligned}$$

while either  $(\mathbf{w}'_{k-1}, \mathbf{w}''_k)$  or  $(\mathbf{w}''_k, \mathbf{w}'_{k-1})$  is a pair of one of the forms (3.1), (3.2). Therefore, it suffices to prove that all ‘endomorphisms’ of pairs (3.1) and (3.2) indeed belong to  $\rho_{a,b}$ . Put differently, for pairs of the form

$$(\mathbf{u}_{a-1} \dots \mathbf{u}_1 \mathbf{x}^2 \mathbf{v}_1 \dots \mathbf{v}_{b-1}, \mathbf{u}_{a-1} \dots \mathbf{u}_1 \mathbf{x} \mathbf{v}_1 \dots \mathbf{v}_{b-1}),$$

$\mathbf{x} \in X^*$ , with the second and the third condition listed after (3.1) and the additional condition that  $c(\mathbf{x}) \subseteq c(\mathbf{u}_1) \cap c(\mathbf{v}_1)$ , and for pairs of the form

$$(\mathbf{u}_{a-1} \dots \mathbf{u}_1 \mathbf{x} \mathbf{y} \mathbf{v}_1 \dots \mathbf{v}_{b-1}, \mathbf{u}_{a-1} \dots \mathbf{u}_1 \mathbf{y} \mathbf{x} \mathbf{v}_1 \dots \mathbf{v}_{b-1}),$$

$\mathbf{x}, \mathbf{y} \in X^*$ , again with the second and the third condition listed after (3.2) and  $c(\mathbf{xy}) \subseteq c(\mathbf{u}_1) \cap c(\mathbf{v}_1)$ , it suffices to prove that they belong to  $\rho_{a,b}$ . However, this is a matter of a routine induction on  $|\mathbf{x}|$  in the first and on  $|\mathbf{xy}|$  in the second case, and is completely analogous to the corresponding result in [1] for the particular case  $b = 1$  (see also [2]).

At this point, let us remark that  $(\mathbf{u}, \mathbf{v}) \in \rho_{1,1}$  if and only if  $c(\mathbf{u}) = c(\mathbf{v})$ ; since one easily verifies that  $\rho_{a,b} \subseteq \rho_{1,1}$  for all  $a, b \geq 1$ , the condition  $(\mathbf{u}, \mathbf{v}) \in \rho_{a,b}$  implies that  $\mathbf{u}$  and  $\mathbf{v}$  have the same content.

**Lemma 7** *Let  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X^*$  be such that  $c(\mathbf{u}), c(\mathbf{v}) \subseteq c(\mathbf{w})$  and  $(\mathbf{u}, \mathbf{v}) \in \rho_{a,b}$  for some  $a, b \geq 1$  satisfying  $a \geq b - 1$ . Then  $(\mathbf{wu}, \mathbf{wv}) \in \rho_{a+1,b}$ .*

*Proof* Consider first the case when  $a \geq b$ . Just as in the previous lemma, if  $(\mathbf{u}, \mathbf{v}) \in \rho_{a,b}$  then there is a sequence of words  $\mathbf{u} = \mathbf{w}_0, \dots, \mathbf{w}_m = \mathbf{v} \in X^*$  such that for each  $1 \leq k \leq m$  there are  $\mathbf{p}_k, \mathbf{q}_k, \mathbf{w}'_{k-1}, \mathbf{w}''_k \in X^*$  with

$$\begin{aligned}\mathbf{w}_{k-1} &= \mathbf{p}_k \mathbf{w}'_{k-1} \mathbf{q}_k, \\ \mathbf{w}_k &= \mathbf{p}_k \mathbf{w}''_k \mathbf{q}_k,\end{aligned}$$

and either  $(\mathbf{w}'_{k-1}, \mathbf{w}''_k)$  or  $(\mathbf{w}''_k, \mathbf{w}'_{k-1})$  is a pair of one of the forms (3.1), (3.2). For the sake of an example, assume that  $(\mathbf{w}'_{k-1}, \mathbf{w}''_k)$  is a pair of type (3.1), so that

$$\begin{aligned}\mathbf{w}'_{k-1} &= \mathbf{u}_{a-1} \dots \mathbf{u}_1 x^2 \mathbf{v}_1 \dots \mathbf{v}_{b-1}, \\ \mathbf{w}''_k &= \mathbf{u}_{a-1} \dots \mathbf{u}_1 x \mathbf{v}_1 \dots \mathbf{v}_{b-1},\end{aligned}$$

with the conditions as specified after (3.1). But then

$$\begin{aligned}\mathbf{w}\mathbf{w}_{k-1} &= [(\mathbf{w}\mathbf{p}_k) \mathbf{u}_{a-1} \dots \mathbf{u}_1 x^2 \mathbf{v}_1 \dots \mathbf{v}_{b-1}] \mathbf{q}_k, \\ \mathbf{w}\mathbf{w}_k &= [(\mathbf{w}\mathbf{p}_k) \mathbf{u}_{a-1} \dots \mathbf{u}_1 x \mathbf{v}_1 \dots \mathbf{v}_{b-1}] \mathbf{q}_k.\end{aligned}$$

So, since  $c(\mathbf{w}\mathbf{p}_k) = c(\mathbf{w}) \supseteq c(\mathbf{w}_{k-1}) \supseteq c(\mathbf{u}_i) \cup c(\mathbf{v}_j)$  for all  $1 \leq i \leq a-1, 1 \leq j \leq b-1$ , we conclude that  $(\mathbf{w}\mathbf{w}_{k-1}, \mathbf{w}\mathbf{w}_k) \in \rho_{a+1,b}$ . This implies  $(\mathbf{wu}, \mathbf{wv}) \in \rho_{a+1,b}$ .

Now assume that  $a = b - 1$ . By definition, the condition  $(\mathbf{u}, \mathbf{v}) \in \rho_{b-1,b}$  is equivalent to  $(\overleftarrow{\mathbf{u}}, \overleftarrow{\mathbf{v}}) \in \rho_{b,b-1}$ , so in a fashion similar to the previous paragraph there is a sequence of words  $\overleftarrow{\mathbf{u}} = \mathbf{w}_0, \dots, \mathbf{w}_m = \overleftarrow{\mathbf{v}} \in X^*$  such that for each  $1 \leq k \leq m$  we have

$$\begin{aligned}\mathbf{w}_{k-1} &= \mathbf{p}_k \mathbf{w}'_{k-1} \mathbf{q}_k, \\ \mathbf{w}_k &= \mathbf{p}_k \mathbf{w}''_k \mathbf{q}_k,\end{aligned}$$

for some  $\mathbf{p}_k, \mathbf{q}_k, \mathbf{w}'_{k-1}, \mathbf{w}''_k \in X^*$  such that either  $(\mathbf{w}'_{k-1}, \mathbf{w}''_k)$  or  $(\mathbf{w}''_k, \mathbf{w}'_{k-1})$  is of one of the forms (3.1), (3.2). Taking again the case when  $(\mathbf{w}'_{k-1}, \mathbf{w}''_k)$  is of type (3.1) as an example, we arrive at

$$\begin{aligned}\mathbf{w}_{k-1} \overleftarrow{\mathbf{w}} &= \mathbf{p}_k [\mathbf{u}_{b-1} \dots \mathbf{u}_1 x^2 \mathbf{v}_1 \dots \mathbf{v}_{b-2} (\mathbf{q}_k \overleftarrow{\mathbf{w}})], \\ \mathbf{w}_k \overleftarrow{\mathbf{w}} &= \mathbf{p}_k [\mathbf{u}_{b-1} \dots \mathbf{u}_1 x \mathbf{v}_1 \dots \mathbf{v}_{b-2} (\mathbf{q}_k \overleftarrow{\mathbf{w}})].\end{aligned}$$

Here we have  $c(\mathbf{q}_k \overleftarrow{\mathbf{w}}) = c(\overleftarrow{\mathbf{w}}) = c(\mathbf{w}) \supseteq c(\mathbf{w}_{k-1}) \supseteq c(\mathbf{u}_i) \cup c(\mathbf{v}_j)$  for all  $1 \leq i, j < b-1$ , while  $c(\mathbf{u}_{b-2} \mathbf{v}_{b-2}) \subseteq c(\mathbf{u}_{b-1})$  holds by the assumptions on pairs of the form (3.1). Hence,  $(\mathbf{w}_{k-1} \overleftarrow{\mathbf{w}}, \mathbf{w}_k \overleftarrow{\mathbf{w}}) \in \rho_{b,b}$ . However, we also have  $(\overleftarrow{\mathbf{w}\mathbf{w}_{k-1}}, \overleftarrow{\mathbf{w}\mathbf{w}_k}) \in \rho_{b,b}$  for all  $1 \leq k \leq m$ , implying  $(\mathbf{wu}, \mathbf{wv}) \in \rho_{b,b}$ , as wanted.

Now we make the key technical observation on which our estimates are based. It falls short of a full characterisation of the relation  $\rho_{a,b}$  (unlike in [11, Theorem 3.2], where the congruences  $\rho_{a,1}$  were completely described); nevertheless, it will be quite sufficient for our purpose.

**Lemma 8** *Let  $\mathbf{u}, \mathbf{v} \in X^*$  and  $a \geq 2, b \geq 1$  be such that  $a \geq b$  and the following conditions hold:*

- (i)  $(s(\mathbf{u}), s(\mathbf{v})) \in \rho_{a,b}$ ,
- (ii)  $\sigma(\mathbf{u}) = \sigma(\mathbf{v})$ ,
- (iii)  $(\bar{s}(\mathbf{u}), \bar{s}(\mathbf{v})) \in \rho_{a-1,b}$ .

*Then  $(\mathbf{u}, \mathbf{v}) \in \rho_{a,b}$ .*

*Proof* First of all, note that for any word  $\mathbf{w}$  we have  $c(\mathbf{w}) = c(s(\mathbf{w})\sigma(\mathbf{w}))$ , so  $c(\bar{s}(\mathbf{w})) \subseteq c(s(\mathbf{w})\sigma(\mathbf{w}))$ . Therefore, the assumption (iii), the inequality  $a-1 \geq b-1$ , and the previous lemma yield

$$(s(\mathbf{u})\sigma(\mathbf{u})\bar{s}(\mathbf{u}), s(\mathbf{u})\sigma(\mathbf{u})\bar{s}(\mathbf{v})) \in \rho_{a,b},$$

while by (ii) we have  $s(\mathbf{u})\sigma(\mathbf{u})\bar{s}(\mathbf{v}) = s(\mathbf{u})\sigma(\mathbf{v})\bar{s}(\mathbf{v})$ . Finally, since  $\rho_{a,b}$  is a congruence, from (i) it follows that

$$(s(\mathbf{u})\sigma(\mathbf{v})\bar{s}(\mathbf{v}), s(\mathbf{v})\sigma(\mathbf{v})\bar{s}(\mathbf{v})) \in \rho_{a,b}$$

holds. By transitivity, this gives  $(\mathbf{u}, \mathbf{v}) = (s(\mathbf{u})\sigma(\mathbf{u})\bar{s}(\mathbf{u}), s(\mathbf{v})\sigma(\mathbf{v})\bar{s}(\mathbf{v})) \in \rho_{a,b}$ , as required.

Since the condition  $(\mathbf{u}, \mathbf{v}) \in \rho_{a,b}$  implies that  $\mathbf{u}$  and  $\mathbf{v}$  have the same content, upon fixing an  $n$ -element subset  $X_n$  of  $X$  we can consistently define

$$F(a, b, X_n) = \{[\mathbf{w}]_{a,b} : c(\mathbf{w}) \subseteq X_n\} = X_n^* / \rho_{a,b},$$

where  $[\mathbf{w}]_{a,b}$  denotes the  $\rho_{a,b}$ -class containing the word  $\mathbf{w}$ . We also define

$$E(a, b, X_n) = \{[\mathbf{w}]_{a,b} : c(\mathbf{w}) = X_n\}.$$

Let  $f_n(a, b) = |F(a, b, X_n)|$  and  $p_n(a, b) = |E(a, b, X_n)|$  and note that these values do not depend (due to Lemma 6) on the actual choice of letters belonging to  $X_n$ , but only on the cardinality  $|X_n| = n$ .

**Proposition 9** *Let  $a, n \geq 2$  and  $b \geq 1$  such that  $a \geq b$ . Then we have*

$$p_n(a, b) \leq np_{n-1}(a, b)f_n(a-1, b). \quad (3.3)$$

*Proof* Let  $C_n$  be the set of all words  $\mathbf{w}$  over  $X$  such that  $c(\mathbf{w}) = X_n$ ; then  $C_n$  is an ideal of  $X_n^*$ . We define  $\nu_n : C_n \rightarrow E(a, b, X_n)$  to be the natural homomorphism given by  $\nu_n(\mathbf{w}) = [\mathbf{w}]_{a,b}$  for all  $\mathbf{w} \in C_n$ . The kernel of this mapping is just the restriction of  $\rho_{a,b}$  to  $C_n$ . On the other hand, define a mapping  $\mu_n : C_n \rightarrow X_n \times F(a, b, X_n) \times F(a-1, b, X_n)$  by

$$\mu_n(\mathbf{w}) = (\sigma(\mathbf{w}), [s(\mathbf{w})]_{a,b}, [\bar{s}(\mathbf{w})]_{a-1,b}).$$

Now, an equivalent way of expressing the statement of Lemma 8 is  $\ker \mu_n \subseteq \ker \nu_n$ ; therefore, the number of equivalence classes of  $\ker \nu_n$  on  $C_n$ , which is, by definition,  $p_n(a, b)$ , is not larger than the number of equivalence classes of  $\ker \mu_n$ , that is,  $|\mu_n(C_n)|$ . Of course, there are precisely  $n$  possible choices for  $\sigma(\mathbf{w})$  and once this letter is fixed, we have  $[s(\mathbf{w})]_{a,b} \in E(a, b, X_n \setminus \sigma(\mathbf{w}))$ , the latter being a set of  $p_{n-1}(a, b)$  elements. Finally, there are  $f_n(a-1, b)$  possible ‘candidates’ for  $[\bar{s}(\mathbf{w})]_{a-1,b}$ . Hence,  $|\mu_n(C_n)| \leq np_{n-1}(a, b)f_n(a-1, b)$ , and the proposition is proved.

*Remark 10* By the very definition of  $\rho_{a,b}$ , we have  $(x^{a+b}, x^{a+b-1}) \in \rho_{a,b}$  for any  $x \in X$ . Therefore,  $p_1(a,b) \leq a+b-1$ . Also, by the already remarked description of  $\rho_{1,1}$ , we have  $p_n(1,1) = 1$  for all  $n \geq 1$ . Finally, notice that by duality we have  $f_n(a,b) = f_n(b,a)$  and  $p_n(a,b) = p_n(b,a)$  for all  $n, a, b \geq 1$ . In particular, we have  $f_n(t, t+1) = f_n(t+1, t)$  for all  $n, t \geq 1$ .

#### 4 Upper bounds for free spectra

As we have seen in Lemma 6,  $\rho_{a,b}$  coincides with the *fully invariant* congruence generated by the pairs (3.1) and (3.2). Therefore, bearing in mind Fact 3, we have that  $X^*/\rho_{t,t}$  is the relatively free monoid of the variety  $\mathcal{U}_t$  on  $X$ . (Similarly, Fact 5 yields that  $X^*/\rho_{t,1}$  is the  $\mathcal{V}_t$ -free monoid on  $X$ .) Consequently,  $f_n(\mathcal{U}_t) = f_n(t,t)$  is the free spectrum of the variety  $\mathcal{U}_t$ .

The fact that  $(\mathbf{u}, \mathbf{v}) \in \rho_{a,b}$  implies  $c(\mathbf{u}) = c(\mathbf{v})$  means that an elementary combinatorial argument suffices to establish a connection, given in the following lemma, between the sequences  $f_n(a,b)$  and  $p_n(a,b)$ . The proof is left to the reader. Note that  $p_0(a,b) = 1$  since the only element of  $E(a,b, \emptyset)$  is the empty word.

**Lemma 11** *Let  $a, b \geq 1$ . For each  $n \geq 0$  we have*

$$f_n(a,b) = \sum_{k=0}^n \binom{n}{k} p_k(a,b).$$

*Hence, if  $\log p_n(a,b) \in \mathcal{O}(n^k)$  for some  $k \geq 1$  then also  $\log f_n(a,b) \in \mathcal{O}(n^k)$ .*

*Proof (Theorem 4)* We prove the assertion  $\log f_n(t,t) \in \mathcal{O}(n^{2t-1})$  by induction on  $t \geq 1$ ; simultaneously, to keep the induction moving, we prove that  $\log f_n(t, t+1) \in \mathcal{O}(n^{2t})$  holds for all  $t \geq 1$ .

For  $t = 1$  we have that  $\mathcal{U}_1$  is just the variety of all semilattice monoids, so  $f_n(\mathcal{U}_1) = 2^n$  and thus  $\log f_n(1,1) \in \mathcal{O}(n)$ . On the other hand,  $\log f_n(1,2) = \log f_n(2,1) = \log f_n(\mathcal{V}_2) \in \mathcal{O}(n^2)$  by the already mentioned result from [11], although this can be deduced directly from the previous lemma and Proposition 9 (for  $a = 2$  and  $b = 1$ ), by telescoping the recurrence

$$\log p_n(2,1) \leq \log p_{n-1}(2,1) + \log n + \log f_n(1,1) = \log p_{n-1}(2,1) + \log n + n.$$

Now assume that we have established  $\log f_n(t,t) \in \mathcal{O}(n^{2t-1})$  and  $\log f_n(t, t+1) \in \mathcal{O}(n^{2t})$  for some  $t \geq 1$ . By applying Proposition 9 for  $a = b = t+1$  we obtain  $p_n(t+1, t+1) \leq n p_{n-1}(t+1, t+1) f_n(t, t+1)$ , an instance of (3.3), and so the inequality

$$\log p_n(t+1, t+1) \leq \log p_{n-1}(t+1, t+1) + \log n + \log f_n(t, t+1).$$

Thus for large enough  $n$  (say, for  $n \geq n_0$ ), there is a real constant  $C > 0$  such that we have

$$\log p_n(t+1, t+1) \leq \log p_{n-1}(t+1, t+1) + Cn^{2t}.$$

By a simple telescoping procedure, we get

$$\log p_n(t+1, t+1) \leq C' + C \sum_{k=n_0}^n k^{2t},$$

for a constant  $C' > 0$ , which means that  $\log p_n(t+1, t+1) \in \mathcal{O}(n^{2t+1})$ . By the previous lemma, we conclude  $\log f_n(t+1, t+1) \in \mathcal{O}(n^{2t+1})$ , as wanted. Similarly, an application of Proposition 9 for  $a = t+2$ ,  $b = t+1$  yields

$$\log p_n(t+1, t+2) = \log p_n(t+2, t+1) \leq \log p_{n-1}(t+2, t+1) + \log n + \log f_n(t+1, t+1).$$

By a telescoping process analogous to the previous one, the asymptotic upper bound for  $\log f_n(t+1, t+1)$  just proved gives the required conclusion that both  $\log p_n(t+1, t+2)$  and  $\log f_n(t+1, t+2)$  belong to the class  $\mathcal{O}(n^{2t+2})$ .

The proof of Theorem 1 is now complete.

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## References

1. Almeida, J.: On iterated semidirect products of finite semilattices. *J. Algebra* **142**, 239–251 (1991)
2. Almeida, J.: *Finite Semigroups and Universal Algebra*. World Scientific, Singapore (1994)
3. Berman, J.: Free spectra gaps and tame congruence types. *Internat. J. Algebra Comput.* **5**, 651–672 (1995)
4. Blanchet-Sadri, F.: On semidirect and two-sided semidirect products of  $\mathcal{J}$ -trivial monoids. *Inform. Théor. Appl.* **30**, 457–482 (1996)
5. Crvenković, S., Ruškuc, N.: Log-linear varieties of semigroups. *Algebra Univers.* **33**, 370–374 (1995)
6. Dolinka, I.: On free spectra of completely regular semigroups and monoids. *J. Pure Appl. Algebra* **213**, 1979–1990 (2009)
7. Dolinka, I.: On free spectra of locally testable semigroup varieties. *Glasgow Math. J.* **53**, 623–629 (2011)
8. Grätzer, G., Kisielewicz, A.: A survey of some open problems on  $p_n$ -sequences and free spectra of algebras and varieties. In: *Universal Algebra and Quasigroup Theory*, Jadwisin, 1989, pp. 57–88. Heldermann Verlag, Berlin (1992)
9. Higman, G.: The orders of relatively free groups. In: *Proc. Internat. Conf. Theory of Groups*, Canberra, 1965, pp. 153–165. Gordon & Breach, New York (1967)
10. Hobby, D., McKenzie, R.: *The Structure of Finite Algebras*. Contemporary Math. Vol. 76. Amer. Math. Soc., Providence, RI (1988)
11. Horváth, G., Kátai-Urbán, K., Pach, P.P., Pluhár, G., Pongrácz, A., Szabó, Cs.: On free algebras in varieties generated by iterated semidirect products of semilattices. *Internat. J. Algebra Comput.* (to appear)
12. Kátai-Urbán, K., Szabó, Cs.: Free spectrum of the variety generated by the five element combinatorial Brandt semigroup. *Semigroup Forum* **73**, 253–260 (2006)
13. Kátai-Urbán, K., Szabó, Cs.: On the free spectrum of the variety generated by the combinatorial completely 0-simple semigroups. *Glasgow Math. J.* **49**, 93–98 (2007)
14. Kearnes, K.A.: Congruence modular varieties with small free spectra. *Algebra Univers.* **42**, 165–181 (1999)
15. Neumann, P.: Some indecomposable varieties of groups. *Quart. J. Math. Oxford (2)* **14**, 46–50 (1963)
16. Pluhár, G., Wood, J.: The free spectra of varieties generated by idempotent semigroups. *Algebra Discrete Math.* **2**, 89–100 (2008)
17. Seif, S.W.: Monoids with sub-log-exponential free spectra. *J. Pure Appl. Algebra* **212**, 1162–1174 (2008)
18. Stiffler, P.: Extensions of the fundamental theorem of finite semigroups. *Adv. Math.* **11**, 159–209 (1973)
19. Straubing, H., Thérien, D.: Weakly iterated block products of finite monoids. In: Rajsbaum, S. (ed.) *Proc. LATIN, Cancún, 2002*. Lecture Notes Comput. Sci., vol. 2286, pp. 91–104. Springer-Verlag, Berlin (2002)
20. Straubing, H., Tesson, P., Thérien, D.: Weakly iterated block products and applications to logic and complexity. *Internat. J. Algebra Comput.* **20**, 319–341 (2010)